The Method of Undetermined Coefficients

This method of solution applies to a very particular and frequently occurring differential equation. The equation must be linear, with constant coefficients and with a non-homogeneous term, *g(x)*, that is a solution of some homogeneous linear D.E., which also has constant coefficients. The D.E. takes the form:

(LCU) $$\sum_{k=0}^{n} a_k y^{(k)} = g_u(x).$$

Note the non-homogeneous term, $g_u(x)$, is simply a linear combination of the two functions, $x^n e^{\alpha x} \sin\beta x$ and $x^n e^{\alpha x} \cos\beta x$, where $n \in W$, and $\alpha, \beta \in R$. {As a later "thought exercise" consider the generalization $\alpha, \beta \in C$ and that of $n \in R$}

Recall that associated with (LCU) is a linear homogeneous equation with constant coefficients:

(HC) $$\sum_{k=0}^{n} a_k y^{(k)} = 0.$$

Furthermore, (HC) will have *n*, easily determinable, linearly independent solutions, which are referred to as a fundamental set of (HC). These are determined indirectly by solving the auxiliary equation

(AE) $$\sum_{k=0}^{n} a_k m^k = 0$$

We will denote the fundamental set by F = $\{y_1, y_2, \ldots, y_n\}$. Now also the arbitrary linear combination $y_c = \sum_{k=1}^{n} c_k y_k$ may be called either the general solution of (HC) or the complementary solution of (LCU).

For example associated with

(LCU4) $y^{(4)} + 4y^{(2)} = x^2 + xe^{-x} + \sin(x/2)$

is the auxiliary equation $m^4 + 4m^2 = 0$ with roots $m = \{0, 0, \pm 2i\}$ which reveals the fundamental set F = $\{1, x, \cos(2x), \sin(2x)\}$ and the complementary solution $y_c = c_1 + c_2 x + c_3 \cos(2x) + c_4 \sin(2x)$.

We now find it convenient to introduce the following.

Definition: Given $g_u(x)$, from (LCU), the initial "basic set", B_0, of this function is the set of functions with unit coefficients, such that all derivatives of $g_u(x)$, including $g_u(x)$ itself, can be written as a linear combination of the elements of the elements from this initial basic set.

For example if as in the above example $g_u(x) = x^2 + xe^{-x} + \sin(x/2)$, then

$$B_0 = \left\{x^2, x, 1, xe^{-x}, e^{-x}, \sin(x/2), \cos(x/2\right\}.$$

Notice that

$$\frac{d(x^2)}{dx} = 2 \cdot x, \quad \frac{d^2(x^2)}{dx^2} = 2 \cdot 1, \quad \frac{d(xe^{-x})}{dx} = 1 \cdot e^{-x} - 1 \cdot xe^{-x}, \text{ and } \frac{d(\sin(x/2))}{dx} = 1/2 \cdot \cos(x/2).$$

Definition: Given an initial basic set, B_0, as above and a fundamental set, F, also as above, we determine a final basic set, B, by first taking all elements of B_0 that are not also in F are placed in a final basic set, B. The remaining elements of B_0 are repeatedly multiplied by *x* until they are unique and no longer repeat elements of F nor repeat an element already in the final basic. These modified elements are then also placed in B. Note the size(B_0) = size(B). Also notice how the initial basic set, B_0, depends only on g(*x*), but the final basic set, B, also depends on the fundamental set .

In the above example: F = $\{1, x, \cos(2x), \sin(2x)\}$. $B_0 = \left\{x^2, x, 1, xe^{-x}, e^{-x}, \sin(x/2), \cos(x/2)\right\}$. So we select the elements $\left\{x^2, xe^{-x}, e^{-x}, \sin(x/2), \cos(x/2)\right\}$, from B_0 which, are not repeated in F. Now the remaining two elements of B_0, $\{x, 1\}$ would be multiplied by x^3 so that they are unique and do not repeat any from F or those already selected from B_0. Thus the final basic set would be:

$$B = \left\{x^4, x^3, x^2, xe^{-x}, e^{-x}, \sin(x/2), \cos(x/2)\right\}.$$

Now a form for a particular solution of (N_U) may be formed as a linear combination of the elements of B. For the given example we have

$$y_p = A\cdot x^4 + B\cdot x^3 + C\cdot x^2 + D\cdot xe^{-x} + E\cdot e^{-x} + F\cdot\sin(x/2) + G\cdot\cos(x/2)$$
$$y_p' = 4A\cdot x^3 + 3B\cdot x^2 + 2C\cdot x - D\cdot xe^{-x} + (D-E)\cdot e^{-x} + (F/2)\cos(x/2) - (G/2)\cdot\sin(x/2)$$
$$y_p'' = 12Ax^2 + 6B\cdot x + 2C + D\cdot xe^{-x} + (E-2D)\cdot e^{-x} - (F/4)\sin(x/2) - (G/4)\cdot\cos(x/2)$$
$$y_p''' = 24A\cdot x + 6B - D\cdot xe^{-x} + (3D-E)\cdot e^{-x} - (F/8)\cos(x/2) + (G/8)\cdot\sin(x/2)$$
$$y_p^{(4)} = 24A + D\cdot xe^{-x} + (E-4D)\cdot e^{-x} + (F/16)\sin(x/2) + (F/16)\cdot\cos(x/2)\,.$$

Substitution into the DE $y^{(4)} + 4y^{(2)} = x^2 + xe^{-x} + \sin(x/2)$

$$\begin{array}{cccccccc}
 & & & 24A & +D\cdot xe^{-x} & +(E-4D)\cdot e^{-x} & +(F/16)\sin(x/2) & +(F/16)\cdot\cos(x/2) \\
+ & 48Ax^2 & +24Bx & +8C & +4D\cdot xe^{-x} & +(4E-8D)\cdot e^{-x} & -F\sin(x/2) & -G\cdot\cos(x/2) \\ \hline
= & x^2 & & & +xe^{-x} & & +\sin(x/2) &
\end{array}$$

Equating coefficients leads to

$48A = 1$, $24B = 0$, $24A + 8C = 0$, $5D=1$, $5E-12D=0$, $-15F/16 = 1$, $-15G/16 = 0$.

So $A = 1/48$, $B = 0$, $C = -1/16$, $D = 1/5$, $E = 12/25$, $F = -16/15$, $G = 0$.

Thus $y_p = (1/48)\cdot x^4 - (1/16)\cdot x^2 + (1/5)\cdot xe^{-x} + (12/25)\cdot e^{-x} - (16/15)\cdot\sin(x/2)$

and the general solution of (LCU4) is given by

$$y = c_1 + c_2 x + c_3\cos(2x) + c_4\sin(2x) + (1/48)\cdot x^4 - (1/16)\cdot x^2 + (1/5)\cdot xe^{-x} + (12/25)\cdot e^{-x} - (16/15)\cdot\sin(x/2)\,.$$

<u>The method of undetermined coefficients for solving</u> (LCU) $\sum_{k=0}^{n} a_k y^{(k)} = g_u(x)$.

1.) Determine the roots of the auxiliary equation, (AE) $\sum_{k=0}^{n} a_k m^k = 0$.
2.) Use these roots to determine the fundamental set of solutions, F.
3.) Form the complementary solution y_c as an arbitrary linear combination of the elements from F.
4.) Determine the basic set, B, for (LCU).
5.) A particular solution, y_p, for (LCU) may now be given as an undetermined linear combination of the elements of B.
6.) y_p, is then differentiated as many times as is required and substituted into (LCU).
7.) Coefficients of the like terms are then equated resulting in an N X N linear system having the N undetermined coefficients as variables.
8.) Solving the system determines the coefficients and hence y_p, is determined.
9.) The general solution of (LCU) is then given as y = $y_c + y_p$

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