### Open Partitions

A partition $S$ of a topological space $X$ is called *open,* if the saturation of each open set is open.

***17:3.*** Prove that a partition is open iff the canonical projection $X \to X/_S$ is an open map.

***17:4.*** Prove that if a set $A$ is saturated with respect to an open partition, then $\operatorname{Int} A$ and $\operatorname{Cl} A$ are also saturated.

***17:C.*** *The quotient space of a second countable space with respect to an open partition is second countable.*

***17:D.*** *The quotient space of a first countable space with respect to an open partition is first countable.*

***17:E.*** *Let $S$ be an open partition of a topological space $X$ and $T$ be an open partition of a topological space $Y$. Denote by $S \times T$ the partition of $X \times Y$ consisting of $A \times B$ with $A \in S$ and $B \in T$. Then the injective factor $X \times Y/_{S \times T} \to X/_S \times Y/_T$ of $\operatorname{pr} \times \operatorname{pr} X \times Y \to X/_S \times Y/_T$ is a homeomorphism.*

## 18. Zoo of Quotient Spaces

### Tool for Identifying a Quotient Space with a Known Space

***18.A.*** *If $f : X \to Y$ is a continuous map of a compact space $X$ onto a Hausdorff space $Y$ then the injective factor $f/_{S(f)} : X/_{S(f)} \to Y$ is a homeomorphism.*

***18.B.*** *The injective factor of a continuous map of a compact space to a Hausdorff one is a topological embedding.*

***18.1.*** Describe explicitly partitions of a segment such that the corresponding quotient spaces are all the connected letters of the alphabet.

***18.2.*** Prove that there exists a partition of a segment $I$ with the quotient space homeomorphic to square $I \times I$.

### Tools for Describing Partitions

Usually an accurate literal description of a partition is cumbersome, but can be shortened and made more understandable. Of course, this requires a more flexible vocabulary with lots of words with almost the same meanings. For instance, the words *factorize* and *pass to a quotient* can be replaced by *attach, glue, identify, contract,* and other words accompanying these ones in everyday life.

Some elements of this language are easy to formalize. For instance, factorization of a space $X$ with respect to a partition consisting of a set $A$

and one-point subsets of the complement of $A$ is called a *contraction* (of the subset $A$ to a point), and the result is denoted by $X/_A$.

> ***18.3.*** Let $A, B \subset X$ comprise a fundamental cover of a topological space $X$. Prove that the quotient map $A/_{A \cap B} \to X/_B$ of the inclusion $A \hookrightarrow X$ is a homeomorphism.

If $A$ and $B$ are disjoint subspaces of a space $X$, and $f : A \to B$ is a homeomorphism then passing to the quotient of the space $X$ by the partition into one-point subsets of the set $X \smallsetminus (A \cup B)$ and two-point sets $\{x, f(x)\}$, where $x \in A$, is called *gluing* or *identifying* (of sets $A$ and $B$ by homeomorphism $f$).

Rather convenient and flexible way for describing partitions is to describe the corresponding equivalence relations. The main advantage of this approach is that, due to transitivity, it suffices to specify only some pairs of equivalent elements: if one states that $x \sim y$ and $y \sim z$ then it is not needed to state $x \sim z$, since this follows.

Hence, a partition is represented by a list of statements of the form $x \sim y$, which are sufficient to recover the equivalence relation. By such a list enclosed into square brackets, we denote the corresponding partition. For example, the quotient of a space $X$ obtained by identifying subsets $A$ and $B$ by a homeomorphism $f : A \to B$ is denoted by $X/_{[a \sim f(a) \text{ for any } a \in A]}$ or just $X/_{[a \sim f(a)]}$.

Some partitions are easy to describe by a picture, especially if the original space can be embedded into plane. In such a case, as in the pictures below, one draws arrows on segments to be identified to show directions which are to be identified.

Below we introduce all these kinds of descriptions for partitions and give examples of their usage, providing simultaneously literal descriptions. The latter are not nice, but they may help to keep the reader confident about the meaning of the new words and, on the other hand, appreciating the improvement the new words bring in.

### Entrance to the Zoo

***18.C.*** Prove that $I/_{[0 \sim 1]}$ is homeomorphic to $S^1$.

In other words, the quotient space of segment $I$ by the partition consisting of $\{0, 1\}$ and $\{a\}$ with $a \in (0, 1)$ is homeomorphic to a circle.

> *18.C.1.* Find a surjective continuous map $I \to S^1$ such that the corresponding partition into preimages of points consists of one-point subsets of the interior of the segment and the pair of boundary points of the segment.

***18.D.*** Prove that $D^n/_{S^{n-1}}$ is homeomorphic to $S^n$.

In *18.D* we deal with the quotient space of ball $D^n$ by the partition into $S^{n-1}$ and one-point subsets of its interior.

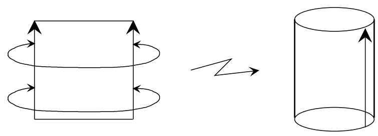
Reformulation of *18.D*: *Contracting* the boundary of an $n$-dimensional ball to a point gives rise to an $n$-dimensional sphere.

> *18.D.1.* Find a continuous map of ball $D^n$ to the sphere $S^n$ that maps the boundary of the ball to a single point, and maps the interior of the ball bijectively onto the complement of this point.

***18.E.*** Prove that $I^2/_{[(0,t)\sim(1,t) \text{ for } t\in \mathrm{I}]}$ is homeomorphic to $S^1\times I$.

Here the partition consisits of pairs of points $\{(0,t),(1,t)\}$ where $t\in I$, and one-point subsets of $(0,1)\times I$.

Reformulation of *18.E*: If we *glue* the side edges of a square identifying points on the same hight, we get a cylinder.



***18.F.*** *Let $X$ and $Y$ be topological spaces, $S$ a partition of $X$. Denote by $T$ the partition of $X\times Y$ into sets $A\times y$ with $A\in S$, $y\in Y$. Then the natural bijection $X/_S\times Y\to X\times Y/_T$ is a homeomorphism.*

***18.G. Riddle.*** How are the problems *18.C*, *18.E* and *18.F* related?

***18.H.*** $S^1\times I/_{[(z,0)\sim(z,1) \text{ for } z\in S^1]}$ is homeomorphic to $S^1\times S^1$.

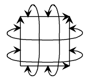
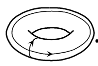
Here the partition consists of one-point subsets of $S^1\times(0,1)$, and pairs of points of the basis circles lying on the same generatrix of the cylinder.

Reformulation of *18.H*: If we *glue* the basis circles of a cylinder identifying points on the same generatrix, then we get a torus.

***18.I.*** $I^2/_{[(0,t)\sim(1,t),\ (t,0)\sim(t,1)]}$ is homeomorphic to $S^1\times S^1$.

In *18.I* the partition consists of

- one-point subsets of the interior $(0,1)\times(0,1)$ of the square,
- pairs of points on the vertical sides, which are the same distance from the bottom side (i.e., pairs $\{(0,t),(1,t)\}$ with $t\in(0,1)$),
- pairs of points on the horizontal sides which lie on the same vertical line (i.e., pairs $\{(t,0),(t,1)\}$ with $t\in(0,1)$),
- the four vertices of the square

Reformulation of *18.I*: Identifying the sides of a square according to the picture  , we get a torus .

## Transitivity of Factorization

A solution of Problem *18.I* can be based on Problems *18.E* and *18.H* and the following general theorem.

***18.J Transitivity of Factorization.*** *Let $S$ be a partition of a space $X$, and let $S'$ be a partition of the space $X/_S$. Then the quotient space $(X/_S)/_{S'}$ is canonically homeomorphic to $X/_T$, where $T$ is the partition of the space $X$ into preimages of elements of the partition $S'$ under projection $X \to X/_S$.*

## Möbius Strip

*Möbius strip* or *Möbius band* is $I^2/_{[(0,t) \sim (1,1-t)]}$. In other words, this is the quotient space of square $I^2$ by the partition into pairs of points symmetric with respect to the center of the square and lying on the vertical edges and one-point set which do not lie on the vertical edges. Figuratively speaking, the Möbius strip is obtained by identifying the vertical sides of a square in such a way that the directions shown on them by arrows are superimposed.

***18.K.*** Prove that the Möbius strip is homeomorphic to the surface swept in $\mathbb{R}^3$ by an interval, which rotates in a halfplane around the middle point while the halfplane rotates around its boundary line. The ratio of the angular velocities of these rotations is such that rotation of the halfplane by 360° takes the same time as rotation of the interval by 180°. See Figure 1.

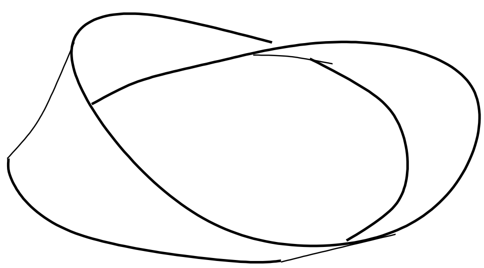


FIGURE 1

### Contracting Subsets

***18.4.*** Prove that $[0,1]/_{[\frac{1}{3},\frac{2}{3}]}$ is homeomorphic to $[0,1]$, and $[0,1]/_{\{\frac{1}{3},1\}}$ is homeomorphic to letter P.

***18.5.*** Prove that the following spaces are homeomorphic:

(a) $\mathbb{R}^2$;
(b) $\mathbb{R}^2/_I$;
(c) $\mathbb{R}^2/_{D^2}$;
(d) $\mathbb{R}^2/_{I^2}$;
(e) $\mathbb{R}^2/_A$ where $A$ is a union of several segments with a common end point;
(f) $\mathbb{R}^2/_B$ where $B$ is a simple finite polygonal line, i.e., a union of a finite sequence of segments $I_1, \dots, I_n$ such that the initial point of $I_{i+1}$ coincides with the final point of $I_i$).

***18.6.*** Prove that if $f : X \to Y$ is a homeomorphism then the quotient spaces $X/_A$ and $Y/_{f(A)}$ are homeomorphic.

***18.7.*** Prove that $\mathbb{R}^2/_{[0,+\infty)}$ is homeomorphic to $\operatorname{Int} D^2 \cup \{(0,1)\}$.

### Further Examples

***18.8.*** Prove that $S^1/_{[z \sim e^{2\pi i/3}z]}$ is homeomorphic to $S^1$.

In *18.8* the partition consists of triples of points which are vertices of equilateral inscribed triangles.

***18.9.*** Prove that the following quotient spaces of disk $D^2$ are homeomorphic to $D^2$:

(a) $D^2/_{[(x,y) \sim (-x,-y)]}$,
(b) $D^2/_{[(x,y) \sim (x,-y)]}$,
(c) $D^2/_{[(x,y) \sim (-y,x)]}$.

***18.10.*** Find a generalization of *18.9* with $D^n$ substituted for $D^2$.

***18.11.*** Describe explicitly the quotient space of line $\mathbb{R}^1$ by equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

***18.12.*** Present the Möbius strip as a quotient space of cylinder $S^1 \times I$.

## Klein Bottle

*Klein bottle* is $I^2/_{[(t,0) \sim (t,1),\ (0,t) \sim (1,1-t)]}$. In other words, this is the quotient space of square $I^2$ by the partition into

- one-point subsets of its interior,
- pairs of points $(t,0), (t,1)$ on horizontal edges which lie on the same vertical line,
- pairs of points $(0,t), (1,1-t)$ symmetric with respect to the center of the square which lie on the vertical edges, and
- the quadruple of vertices.

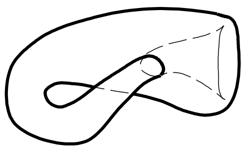
***18.13.*** Present the Klein bottle as a quotient space of

(a) a cylinder;
(b) the Möbius strip.

***18.14.*** Prove that $S^1 \times S^1/[(z, w) \sim (-z, \bar{w})]$ is homeomorphic to the Klein bottle. (Here $\bar{w}$ denotes the complex number conjugate to $w$.)

***18.15.*** Embed the Klein bottle into $\mathbb{R}^4$ (cf. *18.K* and *16.S*).

***18.16.*** Embed the Klein bottle into $\mathbb{R}^4$ so that the image of this embedding under the orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^3$ would look as follows.



## Projective Plane

Let us identify each boundary point of the disk $D^2$ with the antipodal point, i.e., factorize the disk by the partition consisting of one-point subsets of the interior of the disk and pairs of points on the boundary circle symmetric with respect to the center of the disk. The result is called the *projective plane.* This space cannot be embedded into $\mathbb{R}^3$, too. Thus we are not able to draw it. Instead, we present it in other way.

***18.L.*** A projective plane is the result of gluing of a disk and the Möbius strip by homeomorphism between boundary circle of the disk and boundary circle of the Möbius strip.

## You May Have Been Provoked to Perform an Illegal Operation

Solving the previous problem you did something which does not fit into the theory presented above. Indeed, the operation with two spaces called *gluing* in *18.L* has not appeared yet. It is a combination of two operations: first we must make a single space consisting of disjoint copies of the original spaces, and then we factorize this space identifying points of one copy with points of another. Let us consider the first operation in details.

## Set-Theoretic Digression. Sums of Sets

A *sum* of a family of sets $\{X_\alpha\}_{\alpha \in A}$ is the set of pairs $(x_\alpha, \alpha)$ such that $x_\alpha \in X_\alpha$. The sum is denoted by $\coprod_{\alpha \in A} X_\alpha$. The map of $X_\beta$ $(\beta \in A)$ to $\coprod_{\alpha \in A} X_\alpha$ defined by formula $x \mapsto (x, \beta)$ is an injection and denoted by $\mathrm{in}_\beta$. If only sets $X$ and $Y$ are involved and they are distinct, we can avoid indices and define the sum by setting

$$X \amalg Y = \{(x, X) \mid x \in X\} \cup \{(y, Y) \mid y \in Y\}.$$

## Sums of Spaces

***18.M.*** If $\{X_\alpha\}_{\alpha\in A}$ is a collection of topological spaces then the collection of subsets of $\coprod_{\alpha\in A} X_\alpha$ whose preimages under all inclusions $\mathrm{in}_\alpha$ $(\alpha\in A)$ are open, is a topological structure.

The sum $\coprod_{\alpha\in A} X_\alpha$ with this topology is called the *(disjoint) sum of topological spaces* $X_\alpha$, $(\alpha\in A)$.

***18.N.*** Topology described in *18.M* is the finest topology with respect to which all inclusions $\mathrm{in}_\alpha$ are continuous.

***18.17.*** The maps $\mathrm{in}_\beta : X_\beta \to \coprod_{\alpha\in A} X_\alpha$ are topological embedding, and their images are both open and closed in $\coprod_{\alpha\in A} X_\alpha$.

***18.18.*** Which topological properties are inherited from summands $X_\alpha$ by the sum $\coprod_{\alpha\in A} X_\alpha$? Which are not?

## Attaching Space

Let $X$, $Y$ be topological spaces, $A$ a subset of $Y$, and $f : A \to X$ a continuous map. The quotient space $(X \amalg Y)/[a \sim f(a) \text{ for } a\in A]$ is denoted by $X \cup_f Y$, and is said to be the result of *attaching* or *gluing* the space $Y$ to the space $X$ by $f$. The latter is called the *attaching map.*

Here the partition of $X\amalg Y$ consists of one-point subsets of $\mathrm{in}_2(Y \smallsetminus A)$ and $\mathrm{in}_1(X \smallsetminus f(A))$, and sets $\mathrm{in}_1(x)\cup \mathrm{in}_2\big(f^{-1}(x)\big)$ with $x\in f(A)$.

***18.19.*** Prove that the composition of inclusion $X\to X\amalg Y$ and projection $X\amalg Y \to X\cup_f Y$ is a topological embedding.

***18.20.*** Prove that if $X$ is a point then $X\cup_f Y$ is $Y/A$.

***18.O.*** Prove that attaching a ball $D^n$ to its copy by the identity map of the boundary sphere $S^{n-1}$ gives rise to a space homeomorphic to $S^n$.

***18.21.*** Prove that the Klein bottle can be obtained as a result of gluing two copies of the Möbius strip by the identity map of the boundary circle.

***18.22.*** Prove that the result of gluing two copies of a cylinder by the identity map of the boundary circles (of one copy to the boundary circles of the other) is homeomorphic to $S^1\times S^1$.

***18.23.*** Prove that the result of gluing two copies of solid torus $S^1\times D^2$ by the identity map of the boundary torus $S^1\times S^1$ is homeomorphic to $S^1\times S^2$.

***18.24.*** Obtain the Klein bottle by gluing two copies of the cylinder $S^1\times I$ to each other.

***18.25.*** Prove that the result of gluing two copies of solid torus $S^1\times D^2$ by the map

$$S^1\times S^1 \to S^1\times S^1 : (x,y)\mapsto (y,x)$$

of the boundary torus to its copy is homeomorphic to $S^3$.

***18.P.*** Let $X, Y$ be topological spaces, $A$ a subset of $Y$, and $f, g : A \to X$ continuous maps. Prove that if there exists a homeomorphism $h : X \to X$ such that $h \circ f = g$ then $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

***18.Q.*** Prove that $D^n \cup_h D^n$ is homeomorphic to $S^n$ for any homeomorphism $h : S^{n-1} \to S^{n-1}$.

***18.26.*** Classify up to homeomorphism topological spaces, which can be obtained from a square by identifying a pair of opposite sides by a homeomorphism.

***18.27.*** Classify up to homeomorphism the spaces which can be obtained from two copies of $S^1 \times I$ by identifying of the copies of $S^1 \times \{0, 1\}$ by a homeomorphism.

***18.28.*** Prove that the topological type of the space resulting in gluing two copies of the Möbius strip by a homeomorphism of the boundary circle does not depend on the homeomorphism.

***18.29.*** Classify up to homeomorphism topological spaces, which can be obtained from $S^1 \times I$ by identifying $S^1 \times 0$ with $S^1 \times 1$ by a homeomorphism.

## Basic Surfaces

A torus $S^1 \times S^1$ with the interior of an embedded disk deleted is called a *handle*. A two-dimensional sphere with the interior of $n$ disjoint embedded disks deleted is called a *sphere with n holes*.

***18.R.*** A sphere with a hole is homeomorphic to disk $D^2$.

***18.S.*** A sphere with two holes is homeomorphic to cylinder $S^1 \times I$.

A sphere with three holes has a special name. It is called *pantaloons*.

The result of attaching $p$ copies of a handle to a sphere with $p$ holes by embeddings of the boundary circles of handles onto the boundary circles of the holes (the boundaries of the holes) is called a *sphere with p handles*, or, more ceremonial (and less understandable, for a while), *orientable connected closed surface of genus p*.

***18.30.*** Prove that a sphere with $p$ handles is well-defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

***18.T.*** A sphere with one handle is homeomorphic to torus $S^1 \times S^1$.

***18.U.*** A sphere with two handles is homeomorphic to the result of gluing two copies of a handle by the identity map of the boundary circle.

A sphere with two handles is called a *pretzel*. Sometimes this word denotes also a sphere with more handles.