WOULD YOU LIKE SOME PROBABILITY & STATISTISTICS?

(1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

(2) $a + ar + ar^2 + \cdots = \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$ (infinite sum)

(3) $a + ar + ar^2 + \cdots + ar^n = \sum_{i=0}^{n} ar^i = a\left(\frac{1-r^{n+1}}{1-r}\right)$ (finite sum)

(4) $\int_{-\infty}^{\infty} f(x)\,dx = 1$, where $f(x)$ is the density fnc. of a cnts distr.

Geometric

$X \sim \text{geometric}(p) \qquad P(X=x) = p(1-p)^{x-1}$

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1} = pe^t \sum_{x=1}^{\infty} e^{t(x-1)}(1-p)^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} \left[(1-p)e^t\right]^{x-1} = pe^t \sum_{x=0}^{\infty} \left[(1-p)e^t\right]^x$$

$$\overset{2}{=} pe^t \left(\frac{1}{1-(1-p)e^t} \right) = \frac{pe^t}{pe^t - e^t + 1}$$

$$M_X'(0) = \frac{1}{p} = E(X)$$

$$M_X''(0) = \frac{2-p}{p^2}$$

$$\text{Var}\, X = \frac{1-p}{p^2}$$

Another way to calculate $E(X)$:

$$E(X) = \sum_{x=1}^{\infty} x p (1-p)^{x-1} = p \sum_{x=1}^{\infty} x(1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} \frac{-d}{dp}\left[(1-p)^x\right] = -p \frac{d}{dp}\left(\sum_{x=1}^{\infty} (1-p)^x \right)$$

$$\overset{2}{=} -p \frac{d}{dp}\left(\frac{1-p}{1-(1-p)} \right) = -p \frac{d}{dp}\left(\frac{1}{p} - 1 \right) = -p\left(\frac{-1}{p^2} \right) = \frac{1}{p}$$

Gamma

$X \sim \text{gamma}(\alpha, \lambda) \qquad P(X=x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$

$$M_X(t) = \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x}\, dx$$

$\Downarrow$

$\lambda^* = \lambda - t$

Let's make the inside of the integral look like the density fnc of gamma(α, λ^*)

$\Downarrow$

$\frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x}$

$$= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \underbrace{\int_0^\infty \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x}\, dx}_{\text{density of gamma}(\alpha, \lambda-t)\text{ integrates to 1.}} \overset{4}{=} \left(\frac{\lambda}{\lambda-t}\right)^\alpha.$$

$M_X'(0) = \frac{\alpha}{\lambda} = E(X)$

$M_X''(0) = \frac{\alpha(\alpha+1)}{\lambda^2}$

$\Bigg\}\ \text{Var}\, X = \frac{\alpha}{\lambda^2}$

Exponential

$X \sim \text{exponential}(\lambda) = \text{gamma}(1, \lambda)$

$E(X) = \frac{1}{\lambda}$

$\text{Var}\, X = \frac{1}{\lambda^2}$

χ_n^2

$X \sim \chi_n^2 = \text{gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$

$E(X) = n$

$\text{Var}(X) = 2n$

Poisson

$X \sim \text{Poisson}(s) \qquad P(X=x) = \dfrac{e^{-s} s^x}{x!}$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} e^{-s} \frac{s^x}{x!} = e^{-s} \sum_{x=0}^{\infty} \frac{(se^t)^x}{x!}$$

$$\overset{①}{=} e^{-s} e^{se^t} = e^{s(e^t-1)}$$

$$\left.\begin{array}{l} M_X'(0) = s = E(X) \\ M_X''(0) = s+s^2 \end{array}\right\} \quad \text{Var}\, X = s$$

interestingly, $E(X) = \text{Var}\, X$.

Normal

$X \sim \text{Normal}(\mu, \sigma^2)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2} + tx} dx$$

Notice that $\dfrac{-((x-\mu)-\sigma^2 t)^2}{2\sigma^2} = \dfrac{-1}{2\sigma^2}\left((x-\mu)^2 - 2\sigma^2 t(x-\mu) + (\sigma^2 t)^2\right)$

$$= \frac{-(x-\mu)^2}{2\sigma^2} + tx - \mu t - \frac{\sigma^2 t^2}{2}.$$

Hence, $\dfrac{-(x-\mu)^2}{2\sigma^2} + tx = \dfrac{-(x-\mu-\sigma^2 t)^2}{2\sigma^2} + \mu t + \dfrac{\sigma^2 t^2}{2}.$

$$M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu-\sigma^2 t)^2}{2\sigma^2}} e^{\mu t + \frac{\sigma^2 t^2}{2}} dx$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu-\sigma^2 t)^2}{2\sigma^2}}}_{\text{density of Normal}(\mu+\sigma^2 t,\ \sigma^2)\text{; integrates to } 1} dx \overset{④}{=} e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Bernoulli

$X \sim \text{Bernoulli}(p)$

$$P(X=1) = p, \quad P(X=0) = 1-p.$$

$$M_X(t) = \sum_{x=0}^{1} e^{tx} P(X=x) = e^{t\cdot 0} p(X=0) + e^{t\cdot 1} p(X=1)$$

$$= 1-p + pe^t.$$

$$\left.\begin{array}{l} M_X'(0) = p = E(X) \\ M_X''(0) = p \end{array}\right\} \text{Var}\,X = p(1-p).$$

Binomial

$X \sim \text{Binomial}(n,p).$

$X = X_1 + X_2 + \ldots + X_n$, where each $X_i \sim \text{Bernoulli}(p)$ and indep.

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np$$

$$\text{Var}\,X = \text{Var}\left(\sum_{i=1}^{n} X_i\right) \overset{\text{due to}}{\underset{\text{independence}}{=}} \sum_{i=1}^{n} \text{Var}\,X_i = np(1-p).$$

Negative Binomial

$X \sim \text{neg.bin.}(p, j).$

$X = \sum_{i=1}^{j} X_i$ for each $X_i \sim \text{geometric}(p)$ and indep.

$$E(X) = \sum_{i=1}^{j} E(X_i) = \sum_{i=1}^{j} \frac{1}{p} = \frac{j}{p}.$$

$$\text{Var}\,X = \sum_{i=1}^{j} \text{Var}\,X_i = \frac{j(1-p)}{p^2}.$$

due to indep.