Properties of E (E-5 puahaha) w/ PROOF

For X, Y cnts rvs

i) $E(X+Y) = E(X) + E(Y)$

$$\iint (x+y) f_{X,Y}(x,y)\,dy\,dx$$

$$= \iint x f_{X,Y}(x,y)\,dy\,dx + \iint y f_{X,Y}(x,y)\,dy\,dx$$

$$= \int x \int f_{X,Y}(x,y)\,dy\,dx + \int y \int f_{X,Y}(x,y)\,dy\,dx$$

$$= \int x f_X(x)\,dx + \int y f_Y(y)\,dy$$

$$= E(X) + E(Y)$$

For X, Y discrete rvs

$$\sum_{(x,y):P_{X,Y}(x,y)} \sum (x+y) P_{X,Y}(x,y)$$

$$= \sum\sum x P_{X,Y}(x,y) + \sum\sum y P_{X,Y}(x,y)$$

$$= E(X) + E(Y)$$

ii) $E(cX+d) = cE(X) + d$

$$\int (cx+d) f_X(x)\,dx = c \int x f_X(x)\,dx + d \underbrace{\int f_X(x)\,dx}_{1}$$

$$= cE(X) + d$$

$$\sum_x (cx+d) P_X(x)$$

$$= c \sum_x x P_X(x) + d \underbrace{\sum_x P_X(x)}_{1}$$

$$= c\,E(X) + d$$

iii) If X, Y indep.,

$E(XY) = E(X)E(Y)$

$$\iint xy f_{X,Y}(x,y)\,dy\,dx$$

$$= \iint xy f_X(x) f_Y(y)\,dy\,dx \quad \text{(due to indep.)}$$

$$= \int x f_X(x)\,dx \int y f_Y(y)\,dy$$

$$= E(X)E(Y)$$

... similarly ...

Properties of Cov and Var W/ PROOF

$$\text{Cov}(X,Y) = E[(X-\mu_x)(Y-\mu_y)] = E[(X-E(X))(Y-E(Y))].$$

$$\text{Var}\, X = E(X-\mu)^2 \qquad \text{Cov}(X,Y) = E(XY) - \mu_x\mu_y$$

i) $\text{Cov}(X,Y) = \text{Cov}(Y,X)$

$$E[(X-\mu_x)(Y-\mu_y)] = E[(Y-\mu_y)(X-\mu_x)]$$

ii) $\text{Cov}(X,X) = \text{Var}\, X$

$$E[(X-\mu)(X-\mu)] = E(X-\mu)^2$$

iii) $\text{Cov}(aX,Y) = a\,\text{Cov}(X,Y)$

$$E[(aX - E(aX))(Y-\mu_y)] = E[(aX - aE(X))(Y-\mu_y)]$$

$$= E[a(X-\mu_x)(Y-\mu_y)] = aE[(X-\mu_x)(Y-\mu_y)].$$

iv) $\text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) = \sum_{i=1}^n \sum_{j=1}^m \text{Cov}(X_i, Y_j)$

$$E\left[\left(\sum_{i=1}^n (X_i - \mu_{x_i})\right)\left(\sum_{j=1}^m (Y_j - \mu_{y_j})\right)\right] = E\left[\sum_{i=1}^n \sum_{j=1}^m (X_i - \mu_{x_i})(Y_j - \mu_{y_j})\right]$$

$$= \sum_{i=1}^n \sum_{j=1}^m E[(X_i - \mu_{x_i})(Y_j - \mu_{y_j})].$$

v) If X, Y indep $\Rightarrow \text{Cov}(X,Y) = 0$

NOTE: this is a ONE-SIDED ARROW! It doesn't go the other way!

$$E[(X-\mu_x)(Y-\mu_y)] = E[XY - \mu_x Y - \mu_y X - \mu_x\mu_y]$$

$$= E(XY) - \mu_x \underbrace{E(Y)}_{\mu_y} - \mu_y \underbrace{E(X)}_{\mu_x} - \mu_x\mu_y = E(XY) - \mu_x\mu_y$$

indep.

$$= E(X)E(Y) - \mu_x\mu_y = 0.$$

vi) $\text{Var}\, X = E(X^2) - [E(X)]^2$

$$E(X-\mu)^2 = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu \underbrace{E(X)}_{\mu} + \mu^2$$

$$= E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$$

vii) $Var(cX+d) = c^2 \, Var X$

$$E[cX+d - E(cX+d)]^2 = E(cX+d-cE(X)-d)^2 = E[c(X-E(X))]^2$$

$$= c^2 E(X-\mu)^2.$$

viii) If X, Y indep $\Rightarrow Var(X+Y) = Var X + Var Y$

$$E[X+Y-E(X+Y)]^2 = E(X+Y-\mu_x-\mu_y)^2 = E[(X-\mu_x)+(Y-\mu_y)]^2$$

$$= E[(X-\mu_x)^2 + 2(X-\mu_x)(Y-\mu_y) + (Y-\mu_y)^2]$$

$$= E(X-\mu_x)^2 + E(Y-\mu_y)^2 + 2E(X-\mu_x)(Y-\mu_y)$$

$$= Var X + Var Y + 2\,Cov(X,Y).$$ As X, Y indep, by property v: $Cov(X,Y) = 0$.

$$= Var X + Var Y.$$

MGF Properties w/ PROOF

i) $M^{(j)}(0) = E(X^j)$

j^{th} derivative

Cnts case: $\dfrac{d^j M(t)}{dt^j} = \dfrac{d^j}{dt^j} \int e^{tx} f_x(x)\,dx$

$$= \int \frac{d^j}{dt^j}\left(e^{tx} f_x(x)\right) dx = \int x^j e^{tx} f_x(x)\,dx = E(X^j e^{tX})$$

when $t=0$, $M^{(j)}(0) = E(X^j)$. Discrete case is in notes (the same)

ii) $Y = aX+b \Rightarrow M_Y(t) = e^{bt} M_X(at)$

$$M_Y(t) = E[e^{tY}] = E\left(e^{taX} e^{bt}\right) = e^{bt} E\left(e^{t(aX)}\right).$$

iii) X, Y indep. $Z = X+Y \Rightarrow M_Z(t) = M_X(t) M_Y(t)$

$$M_Z(t) = E[e^{tZ}] = E\left(e^{tX} e^{tY}\right) \underset{\text{indep.}}{=} E(e^{tX})\,E(e^{tY})$$

Add 'em up! (Collect 'em all) using this property:

- $\left.\begin{array}{l} X_1 \sim \text{Normal}(\mu_1, \sigma_1^2) \\ X_2 \sim \text{Normal}(\mu_2, \sigma_2^2) \end{array}\right\} X_1 + X_2 \sim \text{Normal}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
- $\left.\begin{array}{l} X_1 \sim \text{Poisson}(s_1) \\ X_2 \sim \text{Poisson}(s_2) \end{array}\right\} X_1 + X_2 \sim \text{Poisson}(s_1 + s_2)$
- $\left.\begin{array}{l} X_1 \sim \text{gamma}(\alpha_1, \lambda) \\ X_2 \sim \text{gamma}(\alpha_2, \lambda) \end{array}\right\} X_1 + X_2 \sim \text{gamma}(\alpha_1 + \alpha_2, \lambda)$
- $\left.\begin{array}{l} X_1 \sim \chi^2_m \\ X_2 \sim \chi^2_n \end{array}\right\} X_1 + X_2 \sim \chi^2_{m+n}$