

Eichler integrals and String Theory*

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Abstract

In this article, it is shown how to obtain objects called *Eichler integrals* in the mathematical literature that can be used for calculating scattering amplitudes in String Theory. These Eichler integrals are also new examples of Eichler integrals with poles.

1 Introduction

The concept of an Eichler integral is closely related to the concept of automorphic forms. Although automorphic forms have a large range of applications in Physics and, in particular in String Theory, Eichler integrals remain relatively unknown objects to both mathematicians and physicists. One can picture an Eichler integral as a generalization of the concept of automorphic forms, and they are related to the better known *Beltrami differentials* that are used in String Theory, in particular in the calculation of multiloop scattering amplitudes of strings.

This article begins with a description of the main properties of automorphic forms with some examples that will be useful when we describe the new Eichler integrals. The definition of Eichler integrals is given next, with some examples that can be used in String Theory.

2 Automorphic Forms

An automorphic form of weight q is a function [1] $\phi(z)$ that transforms in the following way under a projective transformation P_a :

$$\phi(P_a(z)) = \left[\frac{\partial P_a(z)}{\partial z} \right]^q \phi(z) .$$

Automorphic forms with just one pole will be more of our interest, since the order of the pole is limited in a simple way by the Riemann-Roch theorem [2]. We now give some examples of these functions.

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2.1 Example 1

A simple case of an automorphic form of weight q can be given for the case where there is just one projective transformation $P(z)$ given by

$$P(z) = w(z - \alpha) + \alpha ,$$

where α is the finite fixed point of the transformation (the other fixed point is at infinity) and w is the multiplier. An automorphic form $\phi(z)$ of weight q can then be given by

$$\phi(z) = (z - \alpha)^q .$$

In the case where $P(z)$ has two finite fixed points, when it can be expressed by

$$P(z) = \frac{\alpha(x - \beta) - w\beta(z - \alpha)}{(z - \beta) - w(z - \alpha)} ,$$

an automorphic form of weight q is given by

$$\phi(z) = [(z - \alpha)(z - \beta)]^q .$$

2.2 Example 2

In the case of a single projective transformation $P(z)$ with just one fixed point, we may have automorphic forms with poles, given by

$$\phi_n(z) = (z - \alpha)^q \left(\frac{\partial}{\partial t} \right)^{n-1} \mathcal{P}(t) , \quad t = \ln(z - \alpha) - \ln(-\alpha) ,$$

where n is the order of the pole and $\mathcal{P}(t)$ is the Wierstrass \mathcal{P} function [3] with periods $\ln w$ and $2\pi i$.

In the case of $P(z)$ having two finite fixed points, we have

$$\phi(z) = [(z - \alpha)(z - \beta)]^q \left(\frac{\partial}{\partial t} \right)^{n-1} \mathcal{P}(t) , \quad t = \ln \left(\frac{z - \alpha}{z - \beta} \right) - \ln \left(\frac{\alpha}{\beta} \right) ,$$

instead.

2.3 Example 3

Still another way of obtaining an automorphic form on a genus g Riemann surface is considering the following function [1] (called a Poincaré series):

$$\phi(z, \zeta) = \sum_a \left[\frac{dT_a(z)}{dz} \right]^{-q} \frac{\zeta - k}{T_a(z) - \zeta} ,$$

where k is an arbitrary constant and the sum \sum_a is over all elements of the Schottky group, which is the group of all possible combinations of the projective transformations $P_b(z)$, $b = 1, \dots, g$.

Under a transformation $z \rightarrow T_b(z)$, one gets

$$\phi(T_b(z), \zeta) = \sum_a \left[\frac{dT_a T_b(z)}{dz} \right]^{-q} \frac{\zeta - k}{T_a T_b(z) - \zeta} .$$

We now perform a change of variables $T_a T_b(z) = T_c(z)$. Using the chain rule we have

$$\frac{d}{dz} T_c(z) = \left[\frac{d}{dz} T_a T_b(z) \right] \times \frac{d}{dz} T_b(z) ,$$

so that we have

$$\phi(T_b(z), \zeta) = [T'_b(z)]^q \sum_a \left[\frac{dT_c(z)}{dz} \right]^{-q} \frac{\zeta - k}{T_c(z) - \zeta} .$$

Since we are summing over all the elements of the Schottky group, the series above is equivalent to the original one, so that we have

$$\phi(T_b(z), \zeta) = [T'_b(z)]^q \phi(z, \zeta) ,$$

i.e. it is an automorphic form of weight q .

2.4 Example 4

In the multiloop case, let us consider only projective transformations P_a with finite fixed points α_a and β_a . We then consider the following series:

$$P_w(z) = \sum_b \frac{(z - \alpha_b)(z - \beta_b)}{w_b(\alpha_b - \beta_b)} \frac{\delta w_b}{\epsilon} , \quad (1)$$

where the sum is over all the elements of the Schottky group formed by these transformations and δw_b and ϵ are infinitesimals. We then have that

$$\begin{aligned} P_w(T_a(z)) &= \sum_b [T_a(z) - \alpha_b] [T_a(z) - \beta_b] \\ &= \frac{T'_a(z)}{w_a(\alpha_a - \beta_a)^2} \sum_b \frac{1}{w_b(\alpha_b - \beta_b)} (\alpha_a - w_a \beta_a - \alpha_b + w_a \alpha_b) (\alpha_a - w_a \beta_a - \beta_b + w_a \beta_b) \\ &\times \left[z - \frac{(\alpha_a \beta_a - w_a \alpha_a \beta_a - \alpha_b \beta_a + w_a \alpha_b \alpha_a)}{(\alpha_a - w_a \beta_a - \alpha_b + w_a \alpha_b)} \right] \left[z - \frac{(\alpha_a \beta_a - w_a \alpha_a \beta_a - \beta_b \beta_a + w_a \beta_b \alpha_a)}{(\alpha_a - w_a \beta_a - \beta_b + w_a \beta_b)} \right] \frac{\delta w_b}{\epsilon} . \end{aligned}$$

Making then the following change of variables:

$$\alpha_c = \frac{(1 - w_a) \alpha_a \beta_a - (\beta_a - w_a \alpha_a) \alpha_b}{\alpha_a - w_a \beta_a - (1 - w_a) \alpha_b} , \quad (2)$$

$$\beta_c = \frac{(1 - w_a) \alpha_a \beta_a - (\beta_a - w_a \alpha_a) \beta_b}{\alpha_a - w_a \beta_a - (1 - w_a) \beta_b} , \quad (3)$$

$$w_c = w_b , \quad (4)$$

we may write

$$P_w(T_a(z)) = T'_a(z) \sum_c \frac{(z - \alpha_c)(z - \beta_c)}{w_c(\alpha_c - \beta_c)} \frac{\delta w_c}{\epsilon} ,$$

where we have used

$$\delta w_c = \frac{\partial w_c}{\partial w_b} \delta w_b .$$

Since we are summing over all elements of the Schottky group (with the condition that the fixed points are finite), we then see that the expression on the right hand side is equivalent to the series we started with, so that

$$P_w(t_a(z)) = T'_a(z) P_w(z) .$$

We may also consider the following two series:

$$P_\alpha(z) = \sum_b \frac{(1-w_b)}{w_b(\alpha_b - \beta_b)^2} (z - \beta_b)^2 \frac{\delta\alpha_b}{\epsilon}, \quad (5)$$

$$P_\beta(z) = \sum_b \frac{(1-w_b)}{w_b(\alpha_b - \beta_b)^2} (z - \alpha_b)^2 \frac{\delta\beta_b}{\epsilon}. \quad (6)$$

Using the same change of variables (2)-(4) and the fact that

$$\begin{aligned} \delta\alpha_c &= \frac{\partial\alpha_c}{\partial\alpha_b} \delta\alpha_b = \frac{w_a(\alpha_a - \beta_a)^2}{[\alpha_a - w_a\beta_a - (1-w_a)\alpha_b]^2} \delta\alpha_b, \\ \delta\beta_c &= \frac{\partial\beta_c}{\partial\beta_b} \delta\beta_b = \frac{w_a(\alpha_a - \beta_a)^2}{[\alpha_a - w_a\beta_a - (1-w_a)\beta_b]^2} \delta\beta_b, \end{aligned}$$

we then obtain

$$\begin{aligned} P_\alpha(T_a(z)) &= T'_a(z) P_\alpha(z), \\ P_\beta(T_a(z)) &= T'_a(z) P_\beta(z). \end{aligned}$$

We have then obtained three examples of automorphic forms with weight 1.

3 Eichler Integrals

An Eichler integral of order q is defined in the following way [4] [5]: it is a function $f(z)$ that transforms like

$$f(T_a(z)) = \left[\frac{\partial T_a(z)}{\partial z} \right]^q \left[f(z) + P^{q+1}(z) \right],$$

where $P^{q+1}(z)$ is a polynomial at most of order $q+1$. It can be pictured as a generalization of the concept of automorphic form. Eichler integrals are related with automorphic forms in the following way [4]: given an Eichler integral $f(z)$ of order q , we then have

$$\phi(z) = \left(\frac{\partial}{\partial z} \right)^{2q+1} f(z), \quad (7)$$

where $\phi(z)$ is an automorphic form of weight $q+1$, i.e.

$$\phi(T_a(z)) = \left[\frac{\partial T_a(z)}{\partial z} \right]^{q+1} \phi(z).$$

Here are some examples of Eichler integrals.

3.1 Example 1

A trivial example of an Eichler integral is given by any polynomial of order 2, i.e. any function of the form

$$f(z) = a + bz + cz^2,$$

where a , b , and c are constants. Such a function transforms like

$$f(T_a(z)) = T'_a(z) \left[f(z) + \frac{1}{w(\alpha - \beta)^2} (d + ez + fz^2) \right],$$

where

$$\begin{aligned} d &= (1-w)^2 a + (1-w)(\alpha - w\beta)b - \left[w\beta^2(1-w) - 2w(\alpha\beta - 1) + w\alpha^2 \right] c, \\ e &= 2(w\alpha - \beta)(1-w)a - 2\alpha\beta(1-w)^2 b + [2w\beta(\alpha + \beta) - 2\alpha\beta(w\beta + \alpha)] c, \\ f &= (\beta^2 - w\alpha^2)(1-w)a + \alpha\beta(1-w)(\beta - w\alpha)b - w\alpha\beta^2(2 - w\alpha)c. \end{aligned}$$

Differentiating it three times, we obtain

$$\frac{d^3}{dz^3} f(z) = 0,$$

which is a sort of automorphic form (a trivial one) so equation (7) holds.

3.2 Example 2

There are not many examples of Eichler integrals with poles in the literature. The first one was given by Ahlfors [4]. In our notation, his Eichler integral is given by

$$f(z, \zeta) = \sum_b [z - T_b(\zeta)]^{-1} [T'_b(\zeta)]^q, \quad (8)$$

where the sum is over all elements of the Schottky group.

This function transforms in the following way:

$$f(T_a(z), \zeta) = [T'_a(z)]^{q-1} \left\{ f(z, \zeta) + \sum_b \left[\frac{T'_a T_b(\zeta)}{(T'_a(z))^{q-1/2}} - 1 \right] [z - T_b(\zeta)]^{-1} [T'_b(\zeta)]^q \right\}. \quad (9)$$

The second term in the right hand side can be shown to be a polynomial of degree q . This Eichler integral has just one simple pole at $\zeta = z$.

In order to obtain Eichler integrals with poles of higher order, one just has to form derivatives with respect to ζ :

$$f_{k+1}(z, \zeta) = \frac{\partial^k}{\partial \zeta^k} f(z, \zeta)$$

where $f_{k+1}(z, \zeta)$ is an Eichler integral with a pole of order $k + 1$ at $\zeta = z$.

3.3 Example 3

As it will be shown now, not all Eichler integrals must transform in such a complicated way. It is not hard to find examples of Eichler integrals with simple and useful transformations.

As explained before, an Eichler integral can be obtained by simply integrating an automorphic form a certain number of times. It is possible to show that every automorphic form of weight q can be expressed in terms of the following Poincaré series:

$$g(z, \zeta) = \sum_b [T'_b(z)]^{-q} \frac{\zeta - k}{T_b(z) - \zeta}, \quad (10)$$

where k is an arbitrary constant.

If we now take the case $q = -1$ we have simply

$$g(z, \zeta) = \sum_b [T'_b(z)]^{-1} \frac{\zeta - k}{T_b(z) - \zeta} = \sum_b (\zeta - k) \left[\frac{1}{z + (b_b - d_d \zeta)/(a_b - c_b \zeta)} - \frac{1}{z + d_b/c_b} \right],$$

and, by (7), we may obtain an Eichler integral of weight zero simply by integrating this expression once. Doing this we obtain the following:

$$f(z, \zeta) = \sum_b (\zeta - k) \ln \left(\frac{c_b}{(a_b - c_b \zeta)} \frac{[(a_b - c_b \zeta)z + (b_b - d_b \zeta)]}{(c_b z + d_b)} \right) + c' , \quad (11)$$

where c' is a constant of integration that can be set to zero.

We may then make the following change of coefficients:

$$\begin{aligned} a_m &= a_b - c_b \zeta & , & & c_m &= c_b , \\ b_m &= b_b - d_b \zeta & , & & d_m &= d_b , \end{aligned} \quad (12)$$

and the function $f(z, \zeta)$ becomes simply

$$f(z, \zeta) = \sum_m (\zeta - k) \ln \left(\frac{c_m (a_m z + b_m)}{a_m (c_m + d_m)} \right) .$$

Performing the conformal transformation

$$T_a(z) = \frac{a_a z + b_a}{c_a z + d_a} \quad (13)$$

with $c_a \neq 0$ in the expression above, we obtain

$$f(T_a(z), \zeta) = \sum_m (\zeta - k) \ln \left(\frac{c_m [a_m (a_a z + b_a) + b_m (c_a z + d_a)]}{a_m [c_m (a_a z + b_a) + d_m (c_a z + d_a)]} \right) . \quad (14)$$

If we now make another change of variables,

$$\begin{aligned} a_b &= a_m a_a + b_m c_a & , & & c_b &= c_m a_a + d_m c_a , \\ b_b &= a_m b_a + b_m d_a & , & & d_b &= c_m b_a + d_m d_a , \end{aligned} \quad (15)$$

we then have

$$\begin{aligned} f(T_a(z), \zeta) &= \sum_b (\zeta - k) \ln \left(\frac{(c_b d_a - c_a d_b) (a_b z + b_b)}{(a_b d_a - c_a b_b) (c_b z + d_b)} \right) \\ &= \sum_b (\zeta - k) \left[\ln \left(\frac{c_b (a_b z + b_b)}{a_b (c_b z + d_b)} \right) + \ln \left(\frac{a_b (c_b d_a - c_a b_b)}{c_b (a_b d_a - c_a b_b)} \right) \right] . \end{aligned} \quad (16)$$

Since we are summing over all transformations $T_b(z)$ with $c_b \neq 0$, the first term of expression (16) is just $f(z, \zeta)$ and we then have

$$f(T_a(z), \zeta) = f(z, \zeta) + \sum_b (\zeta - k) \ln \left(\frac{a_b (d_b a_a - c_b b_a)}{c_b (b_b a_a - a_b b_a)} \right) . \quad (17)$$

The second term of expression (17) can be readily identified as a constant, i.e. a polynomial in z of degree 0, so that it is proved that the function $f(z, \zeta)$ is an Eichler integral with weight 0.

We now consider automorphic form (10) with weight -2 :

$$g(z, \zeta) = \sum_b [T_b'(z)]^2 \frac{\zeta - k}{T_b(z) - \zeta} , \quad (18)$$

where k is a constant and the sum is over all elements of the Schottky group. According to the theory of Eichler integrals, if we integrate an automorphic form of weight -2 three times we shall obtain an Eichler integral of weight 1.

If we take $T_b(z)$ to be of the form

$$T_b(z) = \frac{a_b z + b_b}{c_b z + d_b}$$

we may then write this function in the following form:

$$g(z, \zeta) = \sum_b \frac{\zeta - k}{a_b - c_b \zeta} \frac{1}{(c_b z + d_b)^3 [z + (b_b - d_b \zeta)/(a_b - c_b \zeta)]}.$$

This can be easily integrated. Considering the case where $c_b \neq 0$ and performing integration three times on this automorphic form we obtain the following function:

$$\begin{aligned} f(z, \zeta) &= \sum_b \frac{1}{2} (\zeta - k) [(a_b - c_b \zeta)z + (b_b - d_b \zeta)]^2 \\ &\quad \times \ln \left(\frac{c_b}{(a_b - c_b \zeta)} \frac{[(a_b - c_b \zeta)z + (b_b - d_b \zeta)]}{(c_b z + d_b)} \right) \\ &\quad + \frac{1}{2c_b(a_b - c_b \zeta)} z - \frac{2(a_b - c_b \zeta)d_b + 1}{4c_b^2(a_b - c_b \zeta)^2} + c_1 z^2 + c_2 z + c_3 \end{aligned} \quad (19)$$

where c_1 , c_2 and c_3 are constants resulting from the integrations. Choosing these integration constants in such a way as to cancel with the polynomial part of expression (19) we then obtain

$$\begin{aligned} f(z, \zeta) &= \sum_b \frac{1}{2} (\zeta - k) [(a_b - c_b \zeta)z + (b_b - d_b \zeta)]^2 \\ &\quad \times \ln \left(\frac{c_b}{(a_b - c_b \zeta)} \frac{[(a_b - c_b \zeta)z + (b_b - d_b \zeta)]}{(c_b z + d_b)} \right). \end{aligned} \quad (20)$$

We can now make the change of coefficients (12) and the function $f(z, \zeta)$ becomes

$$f(z, \zeta) = \sum_m \frac{1}{2} (\zeta - k) (a_m z + b_m)^2 \ln \left(\frac{c_m (a_m z + b_m)}{a_m (c_m z + d_m)} \right). \quad (21)$$

This expression should be an Eichler integral and we are going to show it indeed is. If we perform the conformal transformation

$$T_a(z) = \frac{a_a z + b_a}{c_a z + d_a} \quad (22)$$

with $c_a \neq 0$ in the expression above, we obtain

$$\begin{aligned} f(T_a(z), \zeta) &= \sum_m \frac{1}{2} (\zeta - k) \left[\frac{a_m (a_a z + b_a) + b_m (c_a z + d_a)}{c_a z + d_a} \right]^2 \\ &\quad \times \ln \left(\frac{c_m [a_m (a_a z + b_a) + b_m (c_a z + d_a)]}{a_m [c_m (a_a z + b_a) + d_m (c_a z + d_a)]} \right). \end{aligned} \quad (23)$$

If we now make another change of variables, given by (15), we then have

$$\begin{aligned} f(T_a(z), \zeta) &= T'_a(z) \sum_b \frac{1}{2} (\zeta - k)(a_b z + b_b)^2 \ln \left(\frac{(d_b a_a - c_b b_a)(a_b z + b_b)}{(b_b a_a - a_b b_a)(c_b z + d_b)} \right) \\ &= T'_a(z) \sum_b \frac{1}{2} (\zeta - k)(a_b z + b_b)^2 \left[\ln \left(\frac{c_b(a_b z + b_b)}{a_b(c_b z + d_b)} \right) + \ln \left(\frac{a_b(d_b a_a - c_b b_a)}{c_b(b_b a_a - a_b b_a)} \right) \right] \end{aligned} \quad (24)$$

Since we are summing over all transformations $T_b(z)$ with $c_b \neq 0$, the first term of expression (24) is just $f(z, \zeta)$, and we then have

$$f(T_a(z), \zeta) = T'_a(z) \left[f(z, \zeta) + \sum_b \frac{1}{2} (\zeta - k)(a_b z + b_b)^2 \ln \left(\frac{a_b(d_b a_a - c_b b_a)}{c_b(b_b a_a - a_b b_a)} \right) \right]. \quad (25)$$

The second term of expression (25) can be readily identified with a polynomial in z of degree 2 so that it is proved that the function $f(z, \zeta)$ is an Eichler integral.

4 Example 4

We consider now the case

$$T(z) = \frac{az + d}{cz + d}, \quad (26)$$

that can be written in terms of the finite fixed point α and the multiplier w as

$$T(z) = w(z - \alpha) + \alpha. \quad (27)$$

In String Theory, and in particular in the Group Theoretic approach [6], we will be interested in functions that are associated with conformal transformations that cause infinitesimal changes in the moduli of a Riemann surface with genus g . In one loop, i.e. a Riemann surface with genus 1, we have one finite fixed point α , one fixed point at infinity, and the multiplier w . In order to fix the modular invariance of the theory, we must fix three variables. Besides the one fixed point that has been already fixed at infinity, we can choose to fix the finite fixed point and one of the variables associated with the incoming strings so that the multiplier w will be the only variable associated with the loop left.

We then want a function that has the effect of changing the multiplier w infinitesimally, i.e. we want a function that transforms like

$$f(T(z)) = T'(z) f(z) - \frac{\partial T(z)}{\partial w} \frac{\delta w}{\epsilon}, \quad (28)$$

i.e.

$$f(T(z)) = T'(z) \left[f(z) - \frac{\delta w}{\epsilon w} (z - \alpha) \right] \quad (29)$$

and

$$f(S(z)) = f(z), \quad (30)$$

where $S(z) = e^{2\pi i} z$ and where ϵ and δw are infinitesimals.

A function that transforms in this way was found in references [6] and [7] in the context of String Theory. It is given by

$$f(z) = \frac{\delta w}{\epsilon w} (z - \alpha) \bar{\zeta} (\ln(z - \alpha)), \quad (31)$$

where the function $\bar{\zeta}(t)$ is given by

$$\bar{\zeta}(t) = \zeta(t) - \frac{\zeta(\pi i)}{\pi i} t .$$

The function $\zeta(t)$ is Weierstrass' ζ function,

$$\zeta(t) = \frac{1}{t} + \sum_{p \neq 0} \left(\frac{1}{t-p} + \frac{1}{p} + \frac{1}{p^2} \right) ,$$

where p is the semi-period of the function and is given by $p = nw_1 + mw_2$, ($n, m \in \mathbf{Z}$), where

$$w_1 = \ln w , \quad w_2 = 2\pi i .$$

The Weierstrass ζ function transforms in the following way:

$$\begin{aligned} \zeta(t + w_1) &= \zeta(t) + 2\zeta(w_1/2) , \\ \zeta(t + w_2) &= \zeta(t) + 2\zeta(w_2/2) , \end{aligned}$$

and the term $\ln(z - \alpha)$ transforms like

$$\begin{aligned} \ln(T(z) - \alpha) &= \ln w + \ln(z - \alpha) , \\ \ln(e^{2\pi i} z - e^{2\pi i} \alpha) &= \ln(2\pi i) + \ln(z - \alpha) , \end{aligned}$$

so that we have

$$f(T(z)) = T'(z) \left[f(z) - \frac{\delta w}{\epsilon w} (z - \alpha) \right] . \quad (32)$$

The function $\bar{\zeta}(t)$ can be related to the theta function [3] in the following way:

$$\bar{\zeta}(t) = \frac{d}{dt} \ln \theta(t, \tau) ,$$

where $\theta(t, \tau)$ has periods $w_1 = \ln w$ and $w_2 = 2\pi i$.

If we take the third derivative of the function $f(z)$, we obtain

$$g(z) = f'''(z) = \frac{\delta w}{\epsilon w} \frac{1}{(z - \alpha)^2} \left[\mathcal{P}(t) + \frac{\zeta(\pi i)}{\pi i} - \mathcal{P}''(t) \right] ,$$

where $t = \ln(z - \alpha)$ and $\mathcal{P}(t)$ is Weierstrass' \mathcal{P} function with periods $w_1 = \ln w$ and $w_2 = 2\pi i$, given by

$$\mathcal{P}(t) = -\frac{d}{dt} \zeta(t)$$

which transforms like

$$\mathcal{P}(t + w_1) = \mathcal{P}(t + w_2) = \mathcal{P}(t) ,$$

i.e. it is an elliptic function.

Since $\mathcal{P}(t)$ and its derivatives do not change under a transformation of $t + \ln w$ or $t + 2\pi i$, we then have

$$g(T(z)) = \frac{1}{w^2} g(z) = [T'(z)]^{-2} g(z) ,$$

i.e. $g(z)$ is an automorphic form with weight -2 , as expected from the relation (7). So we have verified that $f(z)$ is an Eichler integral.

4.1 Example 5

We shall now search for a function with transformation properties similar to those of (28), but now for the case of many projective transformations. We want a function that transforms like

$$f_w(T_a(z)) = T'_a(z)f_w(z) - \frac{\partial T_a(z)}{\partial w_a} \frac{\delta w_a}{\epsilon}, \quad (33)$$

where

$$T_a(z) = \frac{\alpha_a(z - \beta_a) - w_a\beta_a(z - \alpha_a)}{(z - \beta_a) - w_a(z - \alpha_a)},$$

for every $a = 1, \dots, g$, i.e. the action of $T_a(z)$ on this function causes infinitesimal changes in the multipliers w_a . So this function must transform like

$$f_w(T_a(z)) = T'_a(z) \left[f_w(z) + \frac{(z - \alpha_a)(z - \beta_a)}{w_a(\alpha_a - \beta_a)} \frac{\delta w_a}{\epsilon} \right]. \quad (34)$$

In addition to this, we also demand that

$$f_a(S_a(z)) = f_a(z), \quad (35)$$

where $z \rightarrow S_a(z)$ is the transformation that takes z once around the a_a loop for $a = 1, \dots, g$.

In order to find the Eichler integral that transforms like this, we shall make analogies between the function in one loop and the function that we must have for the multiloop case. First we notice that the series $P_w(z)$ defined in (1),

$$P_w(z) = \sum_b \frac{(z - \alpha_b)(z - \beta_b)}{w_b(\alpha_b - \beta_b)} \frac{\delta w_b}{\epsilon},$$

transforms like

$$P_w(T_a(z)) = T'_a(z)P_w(z),$$

so that it is the generalization for the multiloop case of the polynomial $(z - \alpha)$ for the one loop case.

Now we must try to find an analog of the Weierstrass ζ function suitable to the multiloop case. This can be obtained by first generalizing the concept of a θ function and of the Weierstrass ζ function. This function is given by the hyperelliptic ζ function [8] [9] or best, by the $\bar{\zeta}$ function defined in the Appendix. In our first attempt we attach a $\bar{\zeta}_b(v)$ function to every element of the series $P_w(z)$, so that we have

$$f_{1w}(z) = \sum_b \frac{\delta w_b}{\epsilon} \frac{(z - \alpha_b)(z - \beta_b)}{w_b(\alpha_b - \beta_b)} \bar{\zeta}_b(v).$$

This function will not transform the way we want, since the term $(z - \alpha_b)(z - \beta_b)/(\alpha_b - \beta_b)$ and the first Abelian integrals $v_b(z)$ do not transform in the same way. We then go to the next step, which is making $\bar{\zeta}$ a function not of the first Abelian integrals $v_b(z)$, but of the variables $u_b(z)$ (such a change of variables is justified in Baker [8], section 192), such that

$$u_b(z) = \ln \left(\frac{z - \alpha_b}{z - \beta_b} \right) - \ln \left(\frac{\alpha_a}{\beta_a} \right).$$

Under a change $z \rightarrow T_a(z)$, these variables will change like

$$u_b(T_a(z)) = \frac{T_a(z) - \alpha_b}{T_a(z) - \beta_b} = \frac{(\alpha_a - w_a\beta_a - \alpha_b + w_a\alpha_b)}{(\alpha_a - w_a\beta_a - \beta_b + w_a\beta_b)} \frac{\left[z - \frac{(\alpha_a\beta_a - w_a\alpha_a\beta_a - \alpha_b\beta_a + \alpha_b w_a\alpha_a)}{(\alpha_a - w_a\beta_a - \alpha_b + w_a\alpha_b)} \right]}{\left[z - \frac{(\alpha_a\beta_a - w_a\alpha_a\beta_a - \beta_b\beta_a + \beta_b w_a\alpha_a)}{(\alpha_a - w_a\beta_a - \beta_b + w_a\beta_b)} \right]} .$$

Performing the same change of variables as in (2)-(4), we then obtain

$$\frac{T_a(z) - \alpha_b}{T_a(z) - \beta_b} = w_{ca} \frac{z - \alpha_c}{z - \beta_c} ,$$

where the coefficient w_{ca} is given by

$$w_{ca} = \frac{(1 - w_a)\beta_c - (\beta_a - w_a\alpha_a)}{(1 - w_a)\alpha_c - (\beta_a - w_a\alpha_a)} ,$$

so that

$$u_b(T_a(z)) = \ln w_{ca} + u_c(z) .$$

We then redefine the generalized θ function $\theta(v)$ in the following way:

$$\theta(u) = \sum_{n=-\infty}^{\infty} \exp \left[\sum_{c,d=1}^g (n_c + \delta_c) \frac{1}{2} \ln w_{cd}(n_d + \delta_d) + \sum_{c=1}^g 2\pi i \gamma_c (n_c + \delta_c) + \sum_{c=1}^g u_c (n_c + \delta_c) \right] .$$

This function transforms like

$$\theta(u + \Omega) = \exp \left\{ - \sum_{c=1}^g p_c \left(u_c + \frac{1}{2} \Omega_c \right) - \sum_{c=1}^g [\pi i p_c q_c - 2\pi i (q_c \delta_c - p_c \gamma_c)] \right\} \theta(u) ,$$

where Ω_b is now given by

$$\Omega_b = \sum_{c=1}^g (2\pi i \delta_{bc} p_c + \ln w_{bc} q_c) = 2\pi i p_b + \sum_{c=1}^g \ln w_{bc} q_c .$$

Defining now

$$\bar{\zeta}_b(u) = \frac{\partial}{\partial u_b} \ln \theta(u) , \tag{36}$$

we have

$$\bar{\zeta}_b(u(T_a(z))) = \bar{\zeta}_b(u) - \delta_{ab}$$

and

$$\bar{\zeta}_b(u(S_a(z))) = \bar{\zeta}_b(u) .$$

We then define the following function:

$$f_w(z) = \sum_b \frac{\delta w_b}{\epsilon} \frac{(z - \alpha_b)(z - \beta_b)}{w_b(\alpha_b - \beta_b)} \bar{\zeta}_b(u(z)) . \tag{37}$$

This function transforms like

$$\begin{aligned} f_w(T_a(z)) &= T'_a(z) \sum_c \frac{\delta w_c}{\epsilon} \frac{(z - \alpha_c)(z - \beta_c)}{w_c(\alpha_c - \beta_c)} [\bar{\zeta}_c(u(z)) - \delta_{ac}] \\ &= T'_a(z) \left[\sum_c \frac{\delta w_c}{\epsilon} \frac{(z - \alpha_c)(z - \beta_c)}{w_c(\alpha_c - \beta_c)} \bar{\zeta}_c(u(z)) - \frac{\delta w_a}{\epsilon} \frac{(z - \alpha_a)(z - \beta_a)}{w_a(\alpha_a - \beta_a)} \right] \end{aligned}$$

and

$$f_w(S_a(z)) = f_w(z) .$$

Since we are summing over all the elements of the Schottky group, we then have

$$\begin{aligned} f_w(T_a(z)) &= T'_a(z) \left[f_w(z) - \frac{\delta w_a}{\epsilon} \frac{(z - \alpha_a)(z - \beta_a)}{w_a(\alpha_a - \beta_a)} \right] , \\ f_w(S_a(z)) &= f_w(z) , \end{aligned}$$

which are the transformation properties we wanted. So we have obtained the Eichler integral that has the effect of changing infinitesimally the variables w_a of a Riemann surface of genus g .

One way to verify this is a true Eichler integral is to take its third derivative. The result must be an automorphic form of weight -2 . Taking the third derivative of (37), we have

$$g_w(z) \equiv \frac{d^3}{dz^3} f_w(z) = \sum_b \frac{\delta w_b}{\epsilon w_b} \left[\frac{(\alpha_b - \beta_b)}{(z - \alpha_b)(z - \beta_b)} \right]^2 [\bar{\mathcal{P}}_{bc}(u(z)) - \bar{\mathcal{P}}''(u(z))] , \quad (38)$$

where

$$\bar{\mathcal{P}}_{bc}(u) = -\frac{\partial}{\partial u_c} \bar{\zeta}_b(u)$$

is the hyperelliptic \mathcal{P} function and

$$\bar{\mathcal{P}}''_{bc}(u) = \frac{\partial^2}{\partial u_c^2} \bar{\mathcal{P}}_{bc}(u_b) .$$

It can be easily verified that $\bar{\mathcal{P}}_{bc}(u)$ and its derivatives are invariant under changes $z \rightarrow T_a(z)$ and $z \rightarrow S_a(z)$, so that the function $g(z)$ transforms like

$$g_w(T_a(z)) = (T'_a(z))^{-2} g_w(z) , \quad (39)$$

i.e. it is an automorphic form of weight -2 . This confirms $f_w(z)$ as an Eichler integral.

4.2 Example 6

In the same way we found a function (Eichler integral) that changes the variables w_a of a Riemann surface infinitesimally we now want to find a function that will change the variables α_a infinitesimally. This means this function must transform like

$$f_\alpha(T_a(z)) = T'_a(z) f_\alpha(z) - \frac{\partial T_a(z)}{\partial \alpha_a} \frac{\delta \alpha_a}{\epsilon} ,$$

for every $a = 1, \dots, g$, what implies that we want this function to transform like

$$f_\alpha(T_a(z)) = T'_a(z) \left[f_\alpha(z) + \frac{(1 - w_a)}{w_a(\alpha_a - \beta_a)^2} (z - \beta_a)^2 \frac{\delta \alpha_a}{\epsilon} \right] . \quad (40)$$

We also demand that

$$f_\alpha(S_a(z)) = f_\alpha(z) .$$

The method to obtain this function is analogous to the one used in the example 5, but now we use, instead of the Poincaré series $P_w(z)$, the series given in (5):

$$P_\alpha(z) = \sum_b \frac{\delta \alpha_b}{\epsilon} \frac{(1 - w_b)}{w_b(\alpha_b - \beta_b)^2} (z - \beta_b)^2 .$$

The function (Eichler integral) with the properties we need is then given by

$$f_\alpha(z) = \sum_b \frac{\delta\alpha_b}{\epsilon} \frac{(1-w_b)}{w_b(\alpha_b - \beta_b)^2} (z - \beta_b)^2 \bar{\zeta}_b(u) . \quad (41)$$

4.3 Example 7

Similarly, if want a function that changes the variables β_a infinitesimally, we then need it to transform like

$$f_\beta(T_a(z)) = T'_a(z) f_\beta(z) - \frac{\partial T_a(z)}{\partial \beta_a} \frac{\delta\beta_a}{\epsilon} ,$$

for every $a = 1, \dots, g$, i.e.

$$f_\beta(T_a(z)) = T'_a(z) \left[f_\beta(z) + \frac{(1-w_a)}{w_a(\alpha_a - \beta_a)^2} (z - \alpha_a)^2 \frac{\delta\beta_a}{\epsilon} \right] . \quad (42)$$

We also demand that

$$f_\beta(S_a(z)) = f_\beta(z) . \quad (43)$$

We then use the Poincaré series (6),

$$P_\beta(z) = \sum_b \frac{\delta\beta_b}{\epsilon} \frac{(1-w_b)}{w_b(\alpha_b - \beta_b)^2} (z - \alpha_b)^2 ,$$

and build the Eichler integral

$$f_\beta(z) = \sum_b \frac{\delta\beta_b}{\epsilon} \frac{(1-w_b)}{w_b(\alpha_b - \beta_b)^2} (z - \alpha_b)^2 \bar{\zeta}_b(u) , \quad (44)$$

which has the correct transformation properties.

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A Some geometrical objects defined on a genus g Riemann surface

A.1 Generalized θ functions

The generalized ϑ function is defined in the following way [8]:

$$\begin{aligned} \vartheta(v; \gamma, \delta) = & \sum_{n=-\infty}^{\infty} \exp \left\{ \sum_{a,b=1}^g \left[\frac{1}{2\pi i} v_a \eta_{(2)ab} v_b + \pi i (n_a + \delta_a) \tau_{ab} (n_b + \delta_b) \right] \right. \\ & \left. + \sum_{a=1}^g [v_a (n_a + \delta_a) + 2\pi i \gamma_a (n_a + \delta_a)] \right\} , \end{aligned}$$

where the sum $\sum_{n=-\infty}^{\infty}$ means the sum over all n_a , $a = 1, \dots, g$. Here, v_a and v_b are the g first Abelian integrals, $\eta_{(2)ab}$ is a symmetric $g \times g$ matrix, n_a and n_b are integers and γ_a and δ_a are vectors with g components that are the characteristics of the function.

If all the elements in the rows p and q are integers, this function transforms like [8]

$$\vartheta(v + \Omega; \gamma, \delta) = \exp \left\{ \sum_{a=1}^g \left[H_a \left(v_a + \frac{1}{2} \Omega_a \right) - \pi i p_a q_a + 2\pi i (q_a \delta_a - p_a \gamma_a) \right] \right\} \vartheta(v; \gamma, \delta)$$

where $v + \Omega$ stands for the sums $v_a + \Omega_a$ ($a = 1, \dots, g$) where

$$\Omega_a = 2\pi i \sum_{b=1}^g \left[2\eta_{(1)ab} p_b + 2\eta_{(2)ab} q_b \right]$$

and

$$H_a = \sum_{b=1}^g \left[2\eta_{(1)ab} p_b + 2\eta_{(2)ab} q_b \right]$$

where $\eta_{(1)}$ and $\eta_{(2)}$ are $g \times g$ matrices and p, q are g -vectors.

We can fix $\eta_{(2)ab} = 0$. This implies

$$\eta_{(1)ab} = -\frac{1}{2} \delta_{ab} .$$

We then have the following function:

$$\theta(v; \gamma, \delta) = \sum_{n=-\infty}^{\infty} \exp \left\{ \sum_{a,b=1}^g \pi i (n_a + \delta_a) \tau_{ab} (n_b + \delta_b) + \sum_{a=1}^g [v_a (n_a + \delta_a) + 2\pi i \gamma_a (n_a + \delta_a)] \right\} ,$$

which transforms like

$$\theta(v + \Omega; \gamma, \delta) = \exp \left\{ \sum_{a=1}^g \left[H_a \left(v_a + \frac{1}{2} \Omega_a \right) - \pi i p_a q_a + 2\pi i (q_a \delta_a - p_a \gamma_a) \right] \right\} \theta(v; \gamma, \delta) .$$

This is the function that is generally referred to as the generalized θ function in the literature.

We may define the well-known object called the *prime form* in terms of the generalized θ function, which is its most general definition. It is given by

$$E(z, \zeta) = \theta(v(z) - v(\zeta)) \left[\sum_{a=1}^g \partial_a \theta(0) w_a(z) \right]^{-1/2} \left[\sum_{b=1}^g \partial_b \theta(0) w_b(z) \right]^{-1/2} ,$$

where

$$\partial_a \theta(0) \equiv \frac{\partial}{\partial v_a} \theta(v) \Big|_{v=0} , \quad \partial_b \theta(0) \equiv \frac{\partial}{\partial v_b} \theta(v) \Big|_{v=0} ,$$

and $w_a(v), w_b(v)$ are first Abelian differentials.

A.2 Hyperelliptic ζ function

We now define the *hyperelliptic ζ -function* [8] [9]:

$$\zeta_a(v) = \frac{\partial}{\partial v_a} \ln \vartheta(v; \gamma, \delta) .$$

This function transforms in the following way:

$$\zeta_a(v + \Omega) = H_a + \zeta_a(v) = \sum_{b=1}^g \left[2\eta_{(1)ab} p_b + 2\eta_{(2)ab} q_b \right] + \zeta_a(v) .$$

The analogy with the one loop ζ -functions is complete when we associate the matrices $\eta_{(1)}$ and $\eta_{(2)}$ with the numbers $\zeta(w_1/2)$ and $\zeta(w_2/2)$, respectively. We then have the identity

$$\sum_{c=1}^g 2\pi i \tau_{ac} \eta_{(2)cb} - 2\pi i \eta_{(1)ab} = \delta_{ab} .$$

A function that will be more useful to us is one that is invariant under a change $\Omega = 2\pi i q_a$ so that we must fix $\eta_{(2)ab} = 0$. Such a function, say $\bar{\zeta}_a$, is defined by

$$\bar{\zeta}_a(v) = \frac{\partial}{\partial v_a} \ln \theta(v; \gamma, \delta)$$

and transforms like

$$\bar{\zeta}_a(v + \Omega) = -p_a + \bar{\zeta}_a(v) .$$

More particularly, this formula shows that the function $\bar{\zeta}$ is invariant under a change $\Omega_a = 2\pi i \tau_{ab} p_b$ for $b \neq a$. It only changes, and then only by a constant term, under the transformation $\Omega_a = 2\pi i \tau_{aa} p_a$ for any p_a .

A.3 Hyperelliptic \mathcal{P} function

We now define the hyperelliptic \mathcal{P} function in the following way [8] [9]:

$$\mathcal{P}_{ab} = \frac{\partial}{\partial v_b} \zeta_a(v) = \frac{\partial}{\partial v_b} \frac{\partial}{\partial v_a} \ln \theta(v) .$$

This function is invariant under changes $\Omega_a = 2\pi i \sum_{b=1}^g \tau_{ab} + 2\pi i q_a$, i.e.

$$\mathcal{P}_{ab}(v + \Omega) = \mathcal{P}_{ab} .$$

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