Galerkin's Method Applied to Mapped Infinite Domains: Quasi-Vector Solutions for Optical Waveguide Modes

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***Abstract*— Galerkin's method applied to mapped infinite domains is employed to analysis the quasi-vector modes of optical waveguide structures. Results are presented for rib waveguides and compare favourably with a near exact Fourier operator transform method.**

I. Introduction

The use of Galerkin's method in solving scalar wave equation for optical waveguides with arbitrary refractive index profiles was first proposed by Henry and Verbeek [1]. Same method was used by Marcuse in solving the vector wave equation [2]. However, there is a large increase in computing time and memory requirement. In considering that matter, we developed a quasi-vector method based on Galerkin's method that include the polarization effects of optical waveguides as an intermediate solution. The memory requirement is the same and the computing time is slightly longer as in solving the scalar wave equation. We also employ the mapping scheme to eliminate the need of enclosing waveguide structures within a rectangle which must be large enough to ensure that the fields of the guided modes of interest are zero at its boundary [3].

II. Mathematical Formulation

For a translationally invariant, real refractive index profile $n(x, y)$, the principal electric field component E_x of the quasi-TE mode, i.e. $E_y = 0$, satisfies the wave equation

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + (n^2k^2 - \beta^2)E_x + 2\frac{\partial}{\partial x}\left[E_x \frac{\partial \ln(n)}{\partial x}\right] = 0 \tag{1}$$

where $E_x = E_x(x, y)$, $n = n(x, y)$ and $k = 2\pi/\lambda$. We map the whole $x-y$ plane onto the unit square in $u-v$ space as in [3] using the transformation functions:

$$x = \alpha_x \tan\left[\pi\left(u - \frac{1}{2}\right)\right] \tag{2a}$$

$$y = \alpha_y \tan\left[\pi\left(v - \frac{1}{2}\right)\right] \tag{2b}$$

where α_x and α_y are arbitrary scaling parameters in the x- and y-directions respectively. By applying the same change of variables, eq. (1) in the transformed coordinate system becomes:

$$\begin{aligned}\left(\frac{du}{dx}\right)^2 \frac{\partial^2 E_x}{\partial u^2} + \frac{d^2u}{dx^2}\frac{\partial E_x}{\partial u} + \left(\frac{dv}{dy}\right)^2 \frac{\partial^2 E_x}{\partial v^2} + \frac{d^2v}{dy^2}\frac{\partial E_x}{\partial v} \\ + \quad \left[n^2k^2 - \beta^2\right]E_x + 2\left(\frac{du}{dx}\right)^2 \frac{\partial}{\partial u}\left[E_x \frac{\partial \ln(n)}{\partial u}\right] \\ + \quad 2\frac{d^2u}{dx^2}E_x \frac{\partial \ln(n)}{\partial u} = 0\end{aligned} \tag{3}$$

where $E_x = E_x(u, v)$ and $n = n(u, v)$. Similar to the scalar solution described in [3], the unknown electric field E_x is expanded as

$$E_x = \sum_{i=1}^{N_m N_n} a_i \phi_i(u, v) = \sum_{m_i=1}^{N_m}\sum_{n_i=1}^{N_n} a_{m_i,n_i}\phi_i(u, v) \tag{4}$$

The expansion functions are chosen as the complete set of orthonormal sinusoidal basis functions

$$\phi_i(u, v) = 2\sin(m_i \pi u)\sin(n_i \pi v) \tag{5}$$

Follow the same procedures in [3], we substituting eq. (4) into eq. (3), multiplying by $\phi_j(u, v)$ and integrating over a unit square to yield the result:

$$\sum_{i=1}^{N_m N_n}\left(S_{j,i} + P_{j,i} - W^2\delta_{j,i}\right)a_i = 0 \tag{6}$$

where

$$S_{j,i} = V^2 A_{j,i} + B_{j,i} \tag{7}$$

corresponding to the scalar wave equation and is detailed in [3, eqs. (7)–(10)]. and

$$P_{j,i} = \rho^2(I_5 + I_6) \tag{8}$$

corresponding to the polarization correction. ρ is a normalisation parameter. The two integrals I_5 and I_6 are given in eqs. (9) and (10).

$$\begin{aligned}I_5 = m_i\pi \int_{u=0}^{1}\int_{v=0}^{1} 2\left(\frac{du}{dx}\right)^2 \frac{1}{\tan(m_i \pi u)} \\ \times\phi_i(u, v)\phi_j(u, v)\frac{\partial \ln(n)}{\partial u}dudv \\ + \int_{u=0}^{1}\int_{v=0}^{1} 2\left(\frac{du}{dx}\right)^2 \phi_i(u, v)\phi_j(u, v)\end{aligned}$$

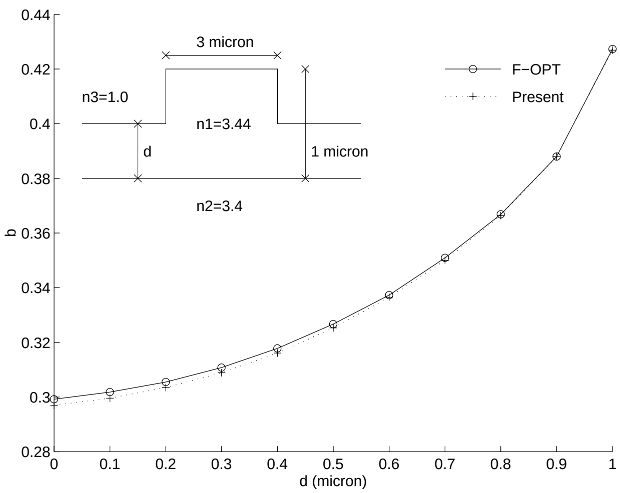

Fig. 1. Normalized propagation constants b for fundamental quasi-TE mode against d. $\lambda = 1.15\mu m$. $N_m^e = N_n = 40$.

$$
\begin{aligned}
&\times \frac{\partial^2 \ln(n)}{\partial u^2} du dv \\
&= \frac{1}{\alpha_x^2} \int_{u=0}^{1} du \int_{v=0}^{1} dv \, \{\ln(n) \\
&\quad \times \{\{16[c(2u) - c(4u)] - m_j^2[c(4u) - 4c(2u) \\
&\quad +3] s_i(u) s_j(u) s_i(v) s_j(v)\} \\
&\quad + m_i m_j [c(4u) - 4c(2u) + 3] c_i(u) c_j(u) s_i(v) s_j(v) \\
&\quad - 4 m_i [s(4u) - 2s(2u)] c_i(u) s_j(u) s_i(v) s_j(v) \\
&\quad - 8 m_j [s(4u) - 2s(2u)] s_i(u) c_j(u) s_i(v) s_j(v)\}\} \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
I_6 &= \int_{u=0}^{1} \int_{v=0}^{1} 2 \left(\frac{d^2 u}{dx^2}\right) \phi_i(u,v) \phi_j(u,v) \frac{\partial \ln(n)}{\partial u} du dv \\
&= \frac{2}{\alpha_x^2} \int_{u=0}^{1} du \int_{v=0}^{1} dv \, \{\ln(n) \\
&\quad \times \{4[c(4u) - c(2u)] s_i(u) s_j(u) s_i(v) s_j(v) \\
&\quad + m_i [s(4u) - 2s(2u)] c_i(u) s_j(u) s_i(v) s_j(v) \\
&\quad + m_j [s(4u) - 2s(2u)] s_i(u) c_j(u) s_i(v) s_j(v)\}\} \qquad (10)
\end{aligned}
$$

The sine and cosine functions have been abbreviated by the symbols s and c respectively. Eq. (6) can be written in matrix form by defining a vector $\mathbf{a}$ consisting of the elements a_i and by also defining a matrix $\mathbf{M}$ composed of the coefficients $S_{j,i}$ and $P_{j,i}$. Equation (6) can now be written in form of a matrix eigenvalue equation with eigenvector $\mathbf{a}$ and eigenvalue W^2.

III. Numerical Results and Discussions

We have applied the present method to study the optical rib waveguide shown in the inset of Fig. 1. A near-exact solution is available for that rib structure by applying a Fourier operator transoform (F-OPT) method [4]. Figure 1 shows the normalised propagation constant $b = [(\beta/k)^2 - n_2^2]/(n_1^2 - n_2^2)$ as a function of d for the fundamental quasi-TE mode. It is observed that present results are slight below the corresponding F-OPT results. Follow the convergence of solutions as show in Table I, results of present method should become much closer to the corresponding F-OPT results if the number of spatial frequency components N_m^e and N_n are increased.

TABLE I

Normalised propagation constant b for quasi-TE modes as calculated by present and F-OPT method for rib waveguide structure shown in inset of Fig. 1. The parameter N_m^e indicate the number of **even** spatial frequency components used in the x-directions and N_n indicate the number of spatial frequency components used in the y-directions.

b	present ($N_m^e = N_n$)			F-OPT
$d(\mu m)$	20	30	40	
0.0	0.2939	0.2963	0.2970	0.2992
0.1	0.2966	0.2990	0.2996	0.3018
0.2	0.3006	0.3029	0.3035	0.3055
0.3	0.3062	0.3084	0.3089	0.3108
0.4	0.3136	0.3156	0.3161	0.3178
0.5	0.3229	0.3248	0.3253	0.3267
0.6	0.3343	0.3361	0.3365	0.3373
0.7	0.3480	0.3496	0.3500	0.3509
0.8	0.3645	0.3661	0.3664	0.3668
0.9	0.3860	0.3875	0.3879	0.3880
1.0	0.4245	0.4264	0.4269	0.4273

IV. Conclusion

We have presented a straightforward, simple, and practical method for calculating the modal propagation constant of quasi-TE mode for optical waveguide structures with arbitrary refractive index profiles. Future research should be continued to include the quasi-TM mode, its formulation is similar to that of quasi-TE mode presented here, and full vector wave solution of optical waveguide structures comprised of arbitrary materials with any shapes or index profiles.

References

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