Galerkin's Method Applied to Mapped Infinite Domains: Vector Solutions for Optical Waveguide Modes

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Agenda

- Introduction
- Mathematical Formulation
- Results and Discussions
- Conclusion

The Method

- In Galerkin's method the vector or scalar field is expressed as a series expansion in terms of a complete set of orthogonal functions.

- Using these field expansions, Galerkin's method is applied to convert equations for transverse field components into an equivalent matrix eigenvalue equation.

Utilization

Year	Author	Regime	Mapping	Publication
1989	Henry & Verbeek	Scalar	No	JLT V7 P308
1992	Marcuse	Vector	No	JQE V28 P459
1995	Hewlett & Ladouceur	Scalar	Yes	JLT V13 P375
1996	Lo & Li	Quasi-vector	Yes	OECC'96 P430
1997	Lo & Li	Vector	Yes	PIERS 1997

Advantages of Mapping

- Artificial boundary is not needed.
- Modal calculations down to cutoff.
- Waveguides with exact inhomogeneous claddings.

Vector Wave Equation for Electric Field

$$\boldsymbol{\nabla}^2\mathbf{E} + n^2k^2\mathbf{E} + \boldsymbol{\nabla}\left[\mathbf{E}\cdot\frac{\boldsymbol{\nabla}n^2}{n^2}\right] = 0$$

- Refractive index $n = n(x, y)$.
- Free-space wavenumber $k = 2\pi/\lambda$.

Two Coupled Equations for E_x and E_y

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + (n^2 k^2 - \beta^2) E_x + 2\frac{\partial}{\partial x}\left[E_x \frac{\partial \ln(n)}{\partial x} + E_y \frac{\partial \ln(n)}{\partial y}\right] = 0$$

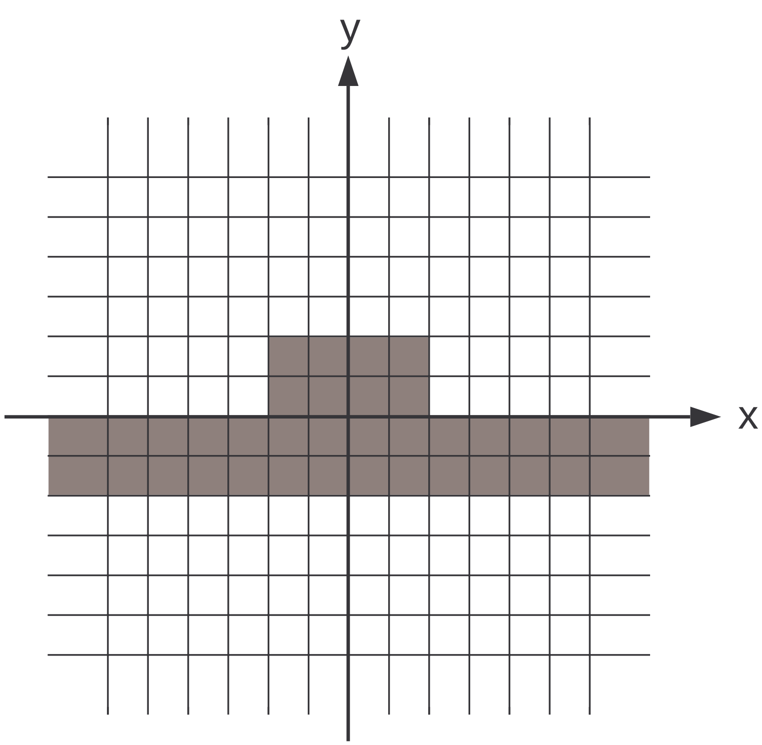
$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + (n^2 k^2 - \beta^2) E_y + 2\frac{\partial}{\partial y}\left[E_x \frac{\partial \ln(n)}{\partial x} + E_y \frac{\partial \ln(n)}{\partial y}\right] = 0$$

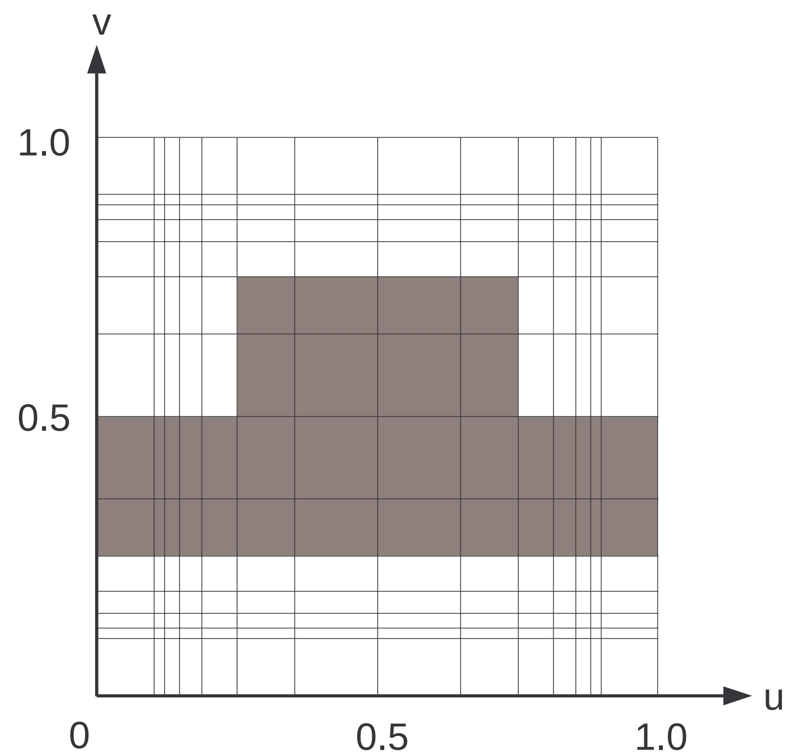
Transformation Functions

$$x = \alpha_x \tan\left[\pi\left(u - \frac{1}{2}\right)\right]$$

$$y = \alpha_y \tan\left[\pi\left(v - \frac{1}{2}\right)\right]$$

Map





Coupled Equations in $u - v$ Space

$$
\begin{aligned}
& \left(\frac{du}{dx}\right)^2 \frac{\partial^2 E_x}{\partial u^2} + \frac{d^2u}{dx^2}\frac{\partial E_x}{\partial u} + \left(\frac{dv}{dy}\right)^2 \frac{\partial^2 E_x}{\partial v^2} \\
+ & \frac{d^2v}{dy^2}\frac{\partial E_x}{\partial v} + \left[n^2k^2 - \beta^2\right] E_x + 2\frac{d^2u}{dx^2}E_x\frac{\partial \ln(n)}{\partial u} \\
+ & 2\left(\frac{du}{dx}\right)^2 \frac{\partial}{\partial u}\left[E_x\frac{\partial \ln(n)}{\partial u}\right] + 2\frac{du}{dx}\frac{dv}{dy}\frac{\partial}{\partial u}\left[E_y\frac{\partial \ln(n)}{\partial v}\right] \\
= & 0
\end{aligned}
$$

Coupled Equations in $u - v$ Space (Cont)

$$
\begin{aligned}
& \left(\frac{du}{dx}\right)^2 \frac{\partial^2 E_y}{\partial u^2} + \frac{d^2u}{dx^2}\frac{\partial E_y}{\partial u} + \left(\frac{dv}{dy}\right)^2 \frac{\partial^2 E_y}{\partial v^2} \\
+ \quad & \frac{d^2v}{dy^2}\frac{\partial E_y}{\partial v} + \left[n^2k^2 - \beta^2\right] E_y + 2\frac{d^2v}{dy^2}E_y\frac{\partial \ln(n)}{\partial v} \\
+ \quad & 2\left(\frac{dv}{dy}\right)^2 \frac{\partial}{\partial v}\left[E_y\frac{\partial \ln(n)}{\partial v}\right] + 2\frac{du}{dx}\frac{dv}{dy}\frac{\partial}{\partial v}\left[E_x\frac{\partial \ln(n)}{\partial u}\right] \\
= \quad & 0
\end{aligned}
$$

Transverse Electric Fields

$$E_x(u,v) = 2\sum_{m_i=1}^{N_m}\sum_{n_i=1}^{N_n} a_{m_i,n_i}\sin(m_i\pi u)\sin(n_i\pi v)$$

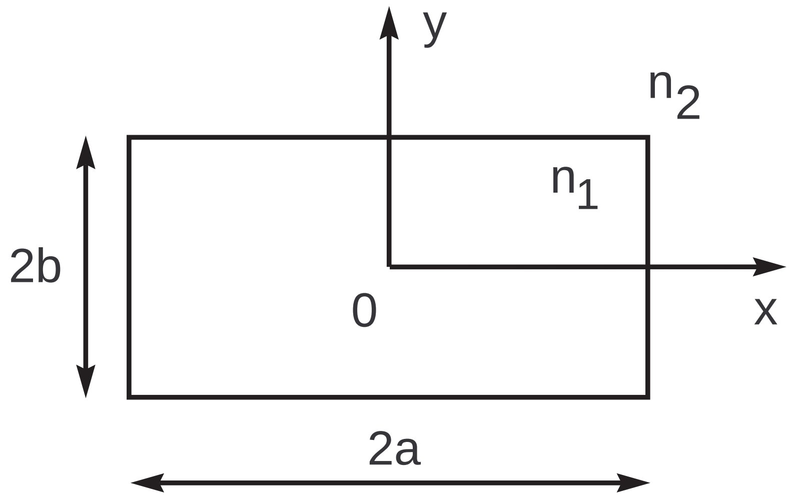
$$E_y(u,v) = 2\sum_{m_i=1}^{N_m}\sum_{n_i=1}^{N_n} b_{m_i,n_i}\sin(m_i\pi u)\sin(n_i\pi v)$$

Matrix Eigenvalue Equation

$$\begin{bmatrix} \mathbf{S} + \mathbf{P}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{S} + \mathbf{P}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = W^2 \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$$

- Modal cladding parameter $W = \rho\sqrt{\beta^2 - k^2 n_{\mathrm{cl}}^2}$.

Numerical Results



- $n_1 = 1.5$, $n_2 = 1.45$, $a = 2b$, $\lambda = 1.15\mu m$.
- $V = kb(n_1^2 - n_2^2)^{1/2}; \qquad P^2 = [(\beta/k)^2 - n_2^2]/(n_1^2 - n_2^2)$.

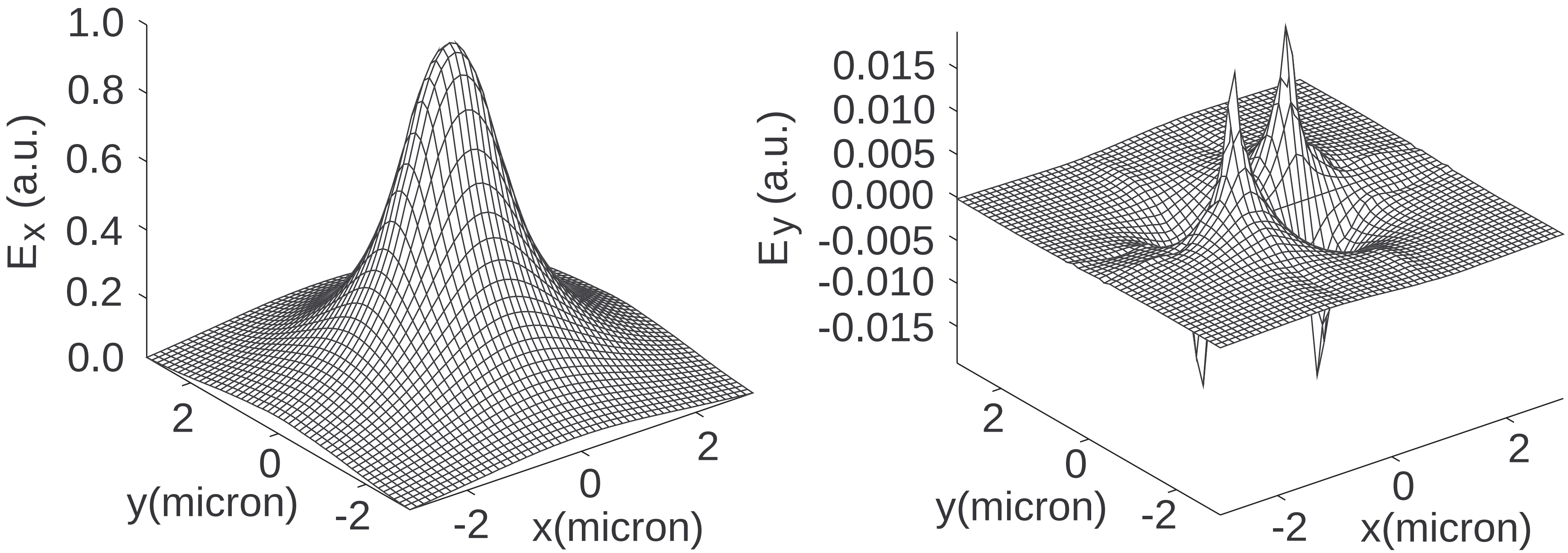
Rectangular Core: $\mathbf{E}^{\mathbf{x}}_{\mathbf{11}}$ mode

V	$N^e_m = N^e_n =$				F-OPT
	10	15	20	25	Quasi-TE
0.6283	0.0346	0.0343	0.0337	0.0335	0.0400
0.7854	0.1076	0.1068	0.1068	0.1069	0.1068
0.9425	0.1990	0.1992	0.1992	0.1991	0.1990
1.0996	0.2909	0.2909	0.2909	0.2908	0.2907
1.3352	0.4117	0.4117	0.4117	0.4117	0.4116
1.5708	0.5090	0.5091	0.5090	0.5090	0.5089

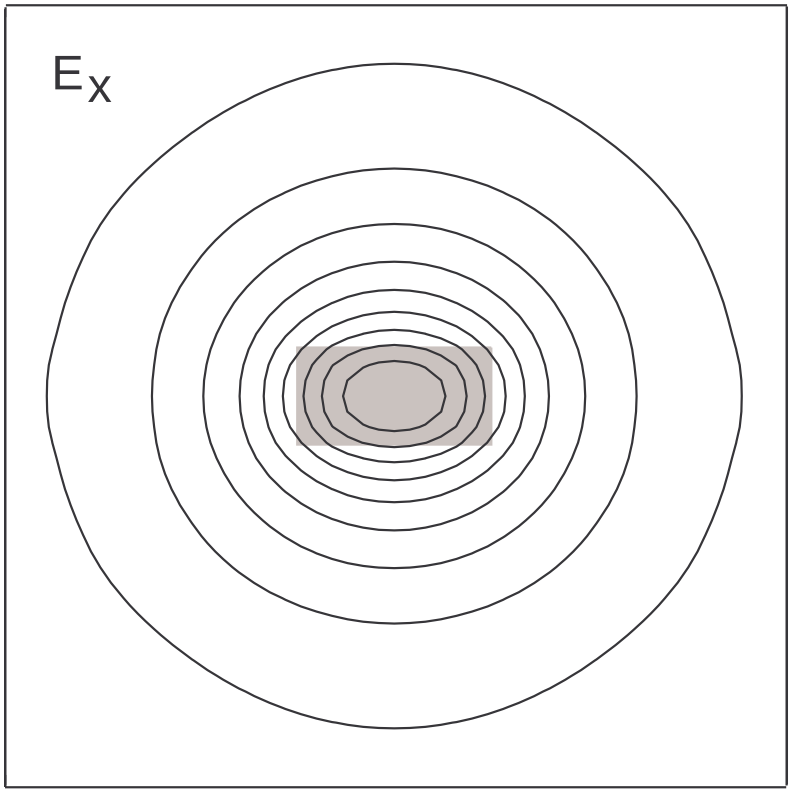
Rectangular Core: $\mathbf{E}^{\mathrm{y}}_{11}$ mode

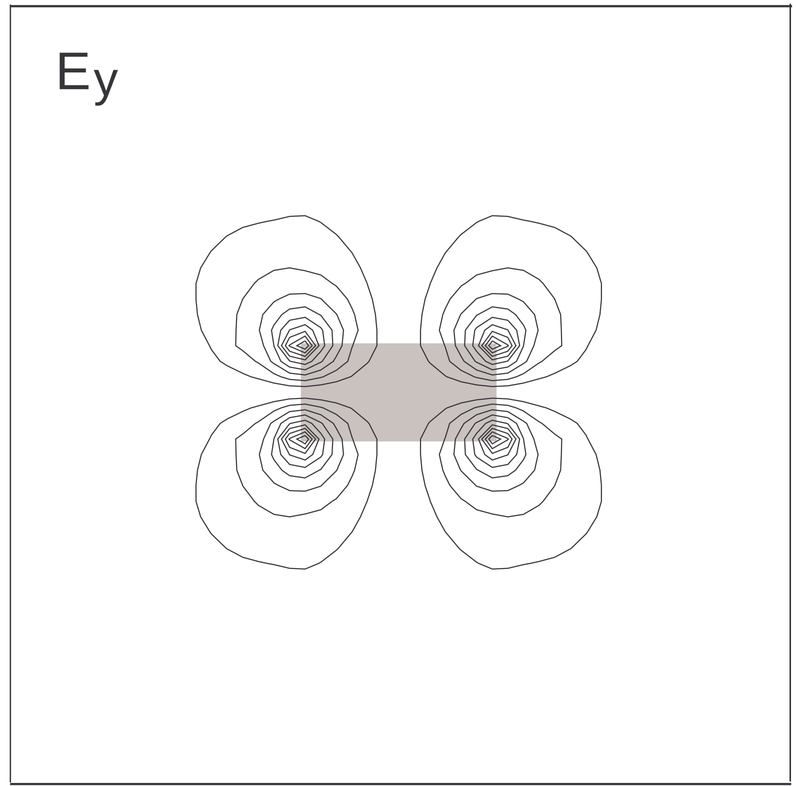
V	$N^o_m = N^o_n =$				F-OPT
	10	15	20	25	Quasi-TM
0.6283	0.0273	0.0315	0.0320	0.0306	0.0483
0.7854	0.0999	0.1004	0.1000	0.1005	0.1003
0.9425	0.1907	0.1903	0.1900	0.1902	0.1900
1.0996	0.2817	0.2810	0.2809	0.2808	0.2805
1.3352	0.4021	0.4017	0.4017	0.4016	0.4013
1.5708	0.5001	0.5000	0.4999	0.4999	0.4996

Rectangular Core: $\mathbf{E}_{11}^{\mathbf{x}}$ mode



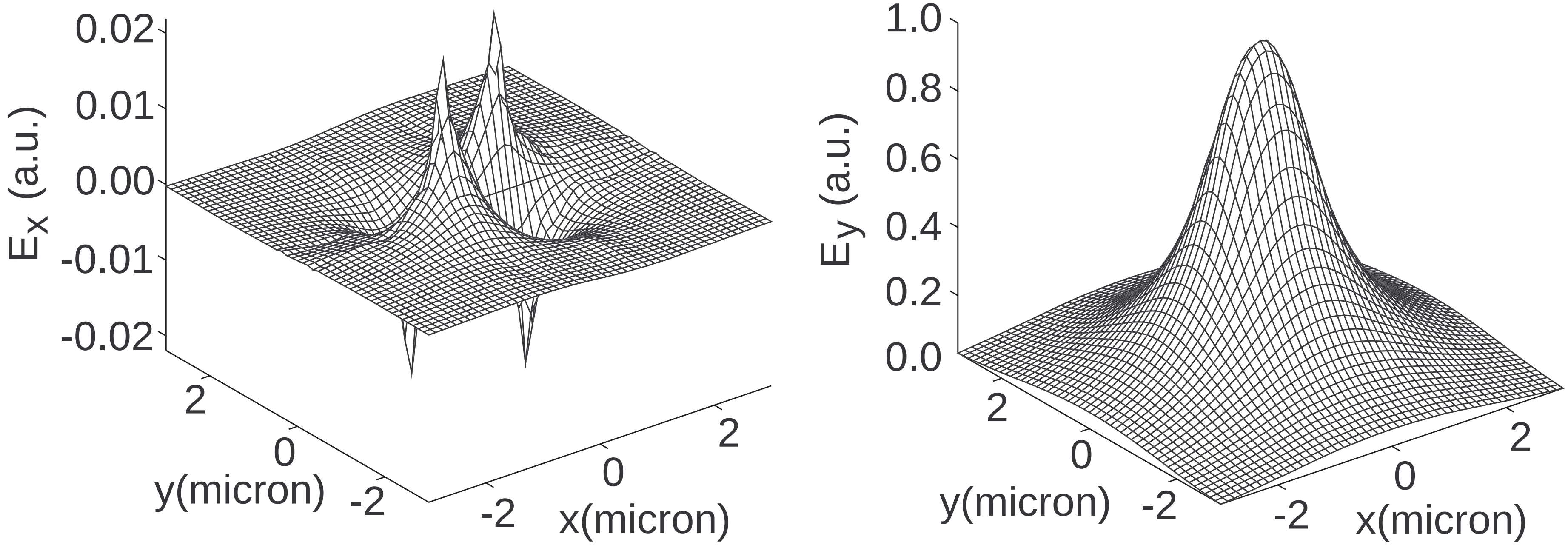
Rectangular Core: $\mathbf{E}^{\mathbf{x}}_{11}$ mode



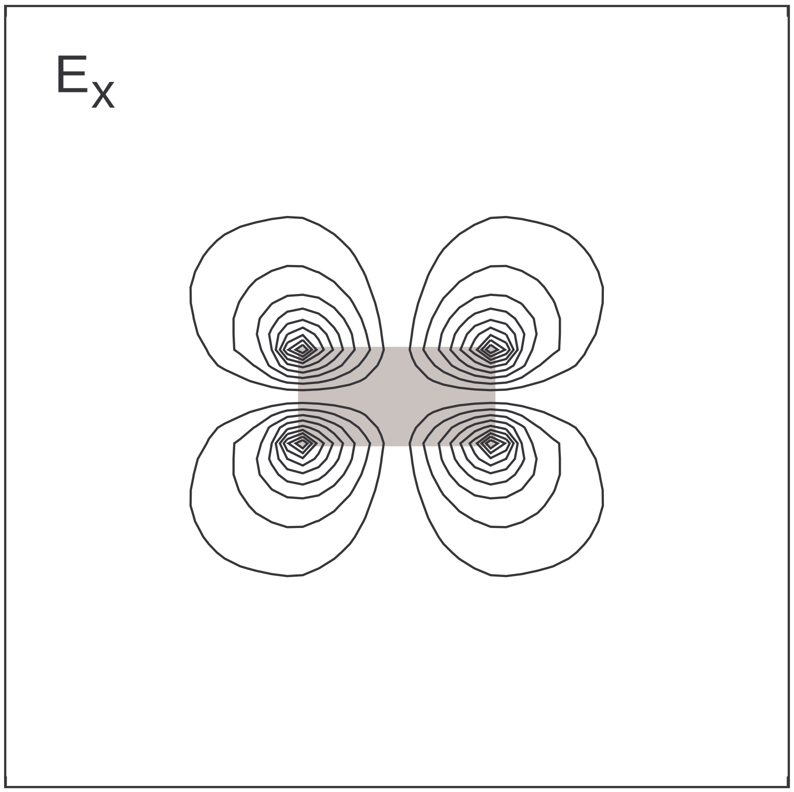


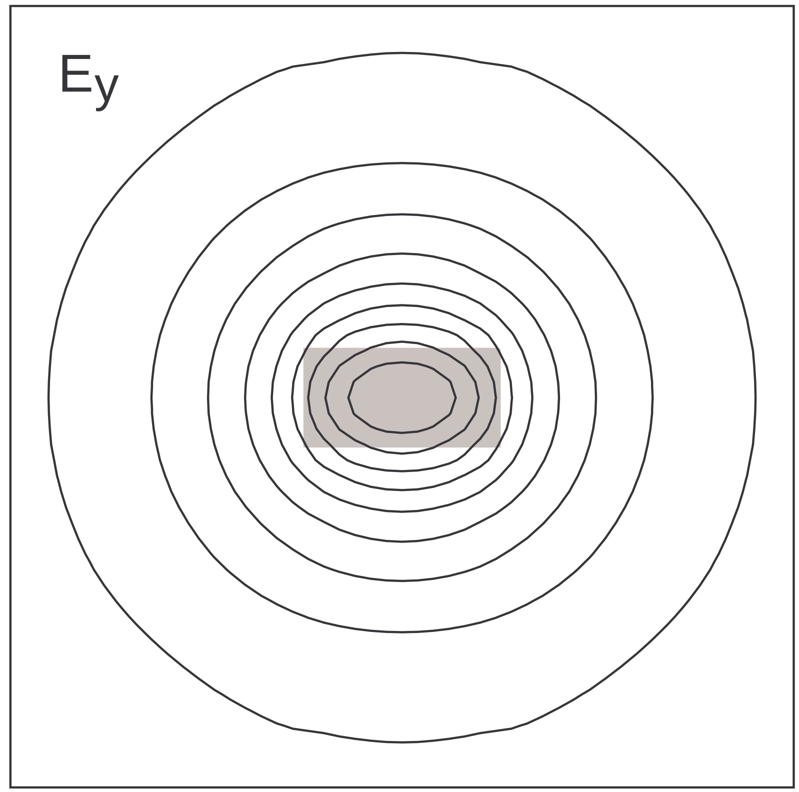
$V = 0.7854$, $N^e_m = N^e_n = 15$

Rectangular Core: E_{11}^{y} mode



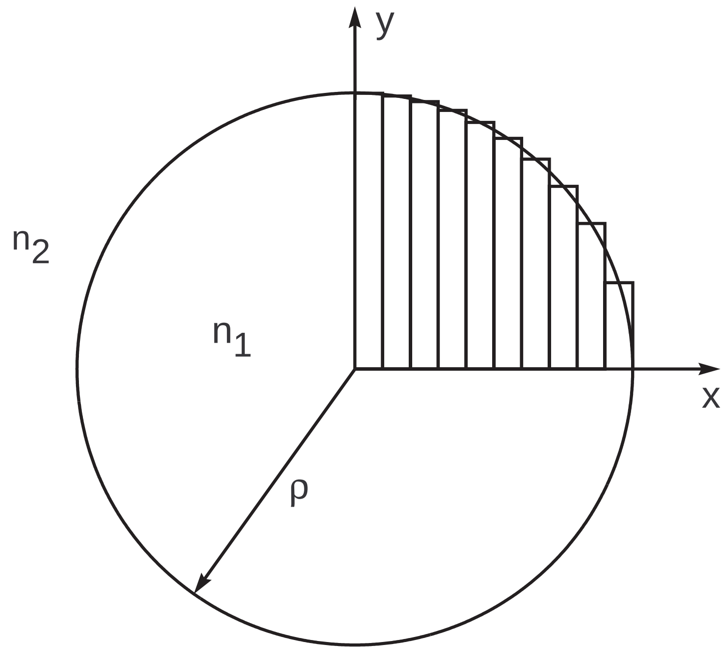
Rectangular Core: $\mathbf{E}^{\mathbf{y}}_{11}$ mode





$V = 0.7854$, $N^o_m = N^o_n = 15$

Circular Core

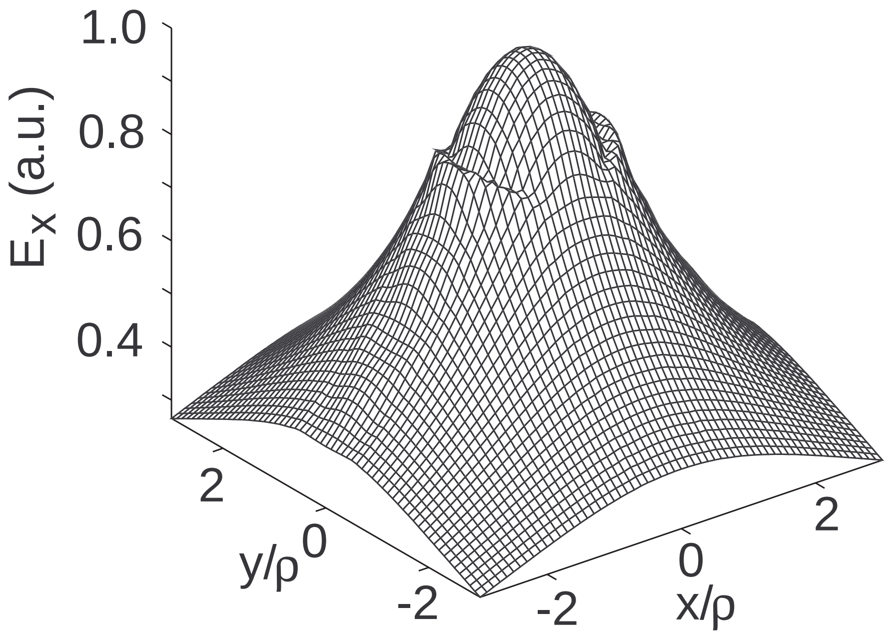


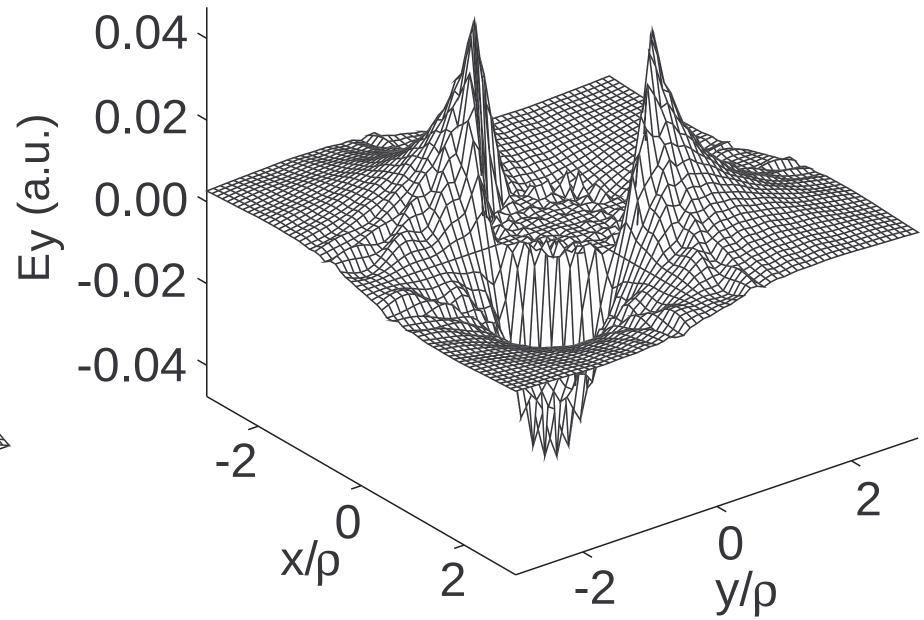
- $n_1 = 1.6$, $n_2 = 1.5$.
- $V = k\rho(n_1^2 - n_2^2)^{1/2}; \qquad P^2 = [(\beta/k)^2 - n_2^2]/(n_1^2 - n_2^2).$

Circular Core: $\mathbf{E}^{\mathbf{x}}_{\mathbf{11}}$ mode

V	$N^e_m = N^e_n =$				Exact
	10	15	20	25	HE_{11}
0.8	-	0.0043	0.0048	0.0046	0.0043
1.0	0.0338	0.0328	0.0324	0.0324	0.0322
1.2	0.0918	0.0916	0.0916	0.0916	0.0911
1.5	0.2102	0.2098	0.2096	0.2095	0.2088
2.0	0.3989	0.3986	0.3985	0.3983	0.3976
2.5	0.5409	0.5407	0.5406	0.5405	0.5399
3.0	0.6419	0.6418	0.6418	0.6417	0.6412

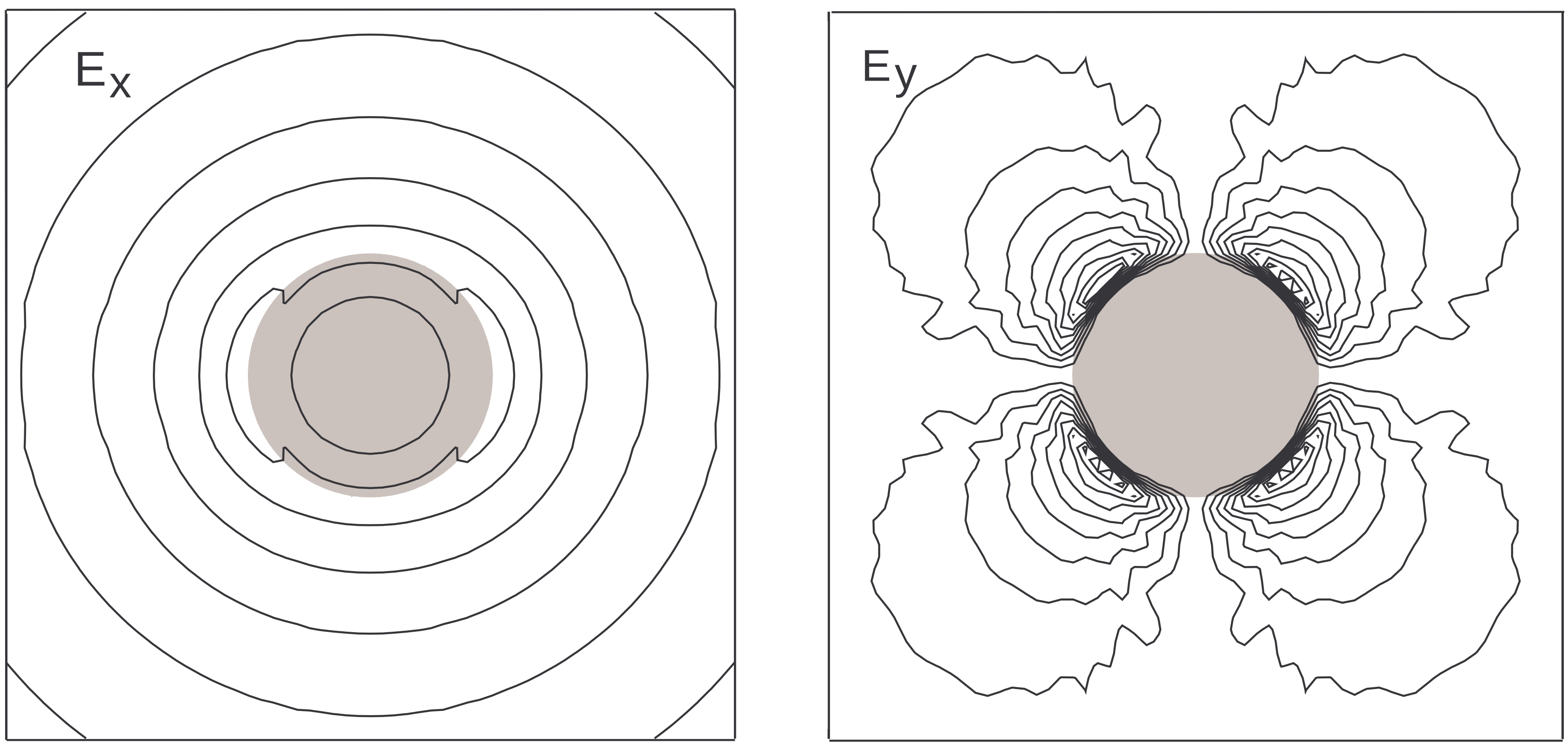
Circular Core: $\mathbf{E}_{11}^{\mathbf{x}}$ mode





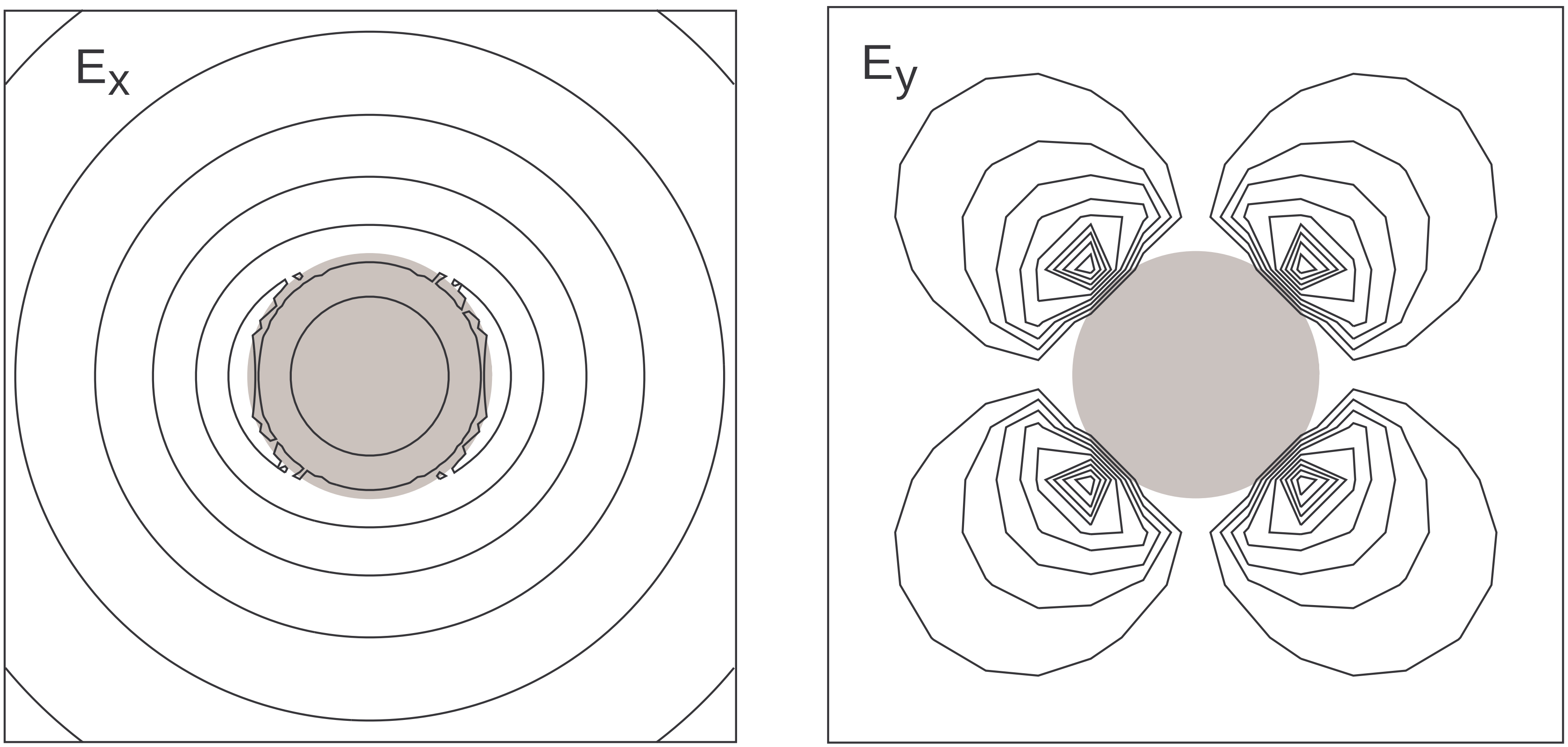
$V = 1.0$, $N_m^e = N_n^e = 20$

Circular Core: $\mathbf{E}_{11}^{\mathbf{x}}$ mode



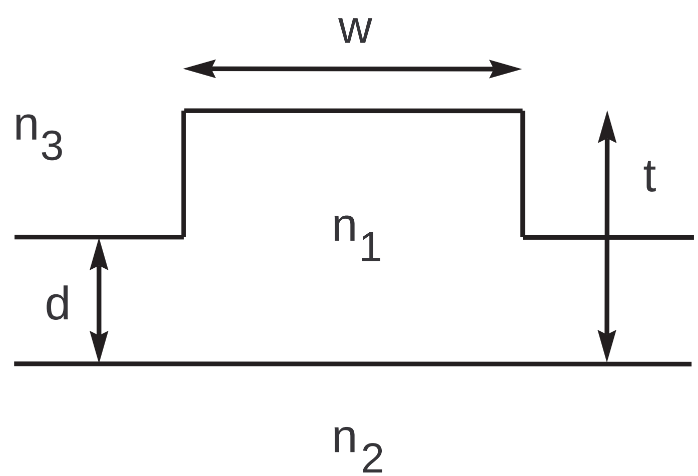
$V = 1.0$, $N_m^e = N_n^e = 20$

Circular Core: Exact HE_{11} mode



$V = 1.0$, Exact vector solution

Rib Waveguide



- $n_1 = 3.44$, $n_2 = 3.4$, $n_3 = 1.0$, $w = 3.0\mu m$, $t = 1.0\mu m$.
- $0.0\mu m \leq d \leq 1.0\mu m$; $\quad P^2 = [(\beta/k)^2 - n_2^2]/(n_1^2 - n_2^2)$.

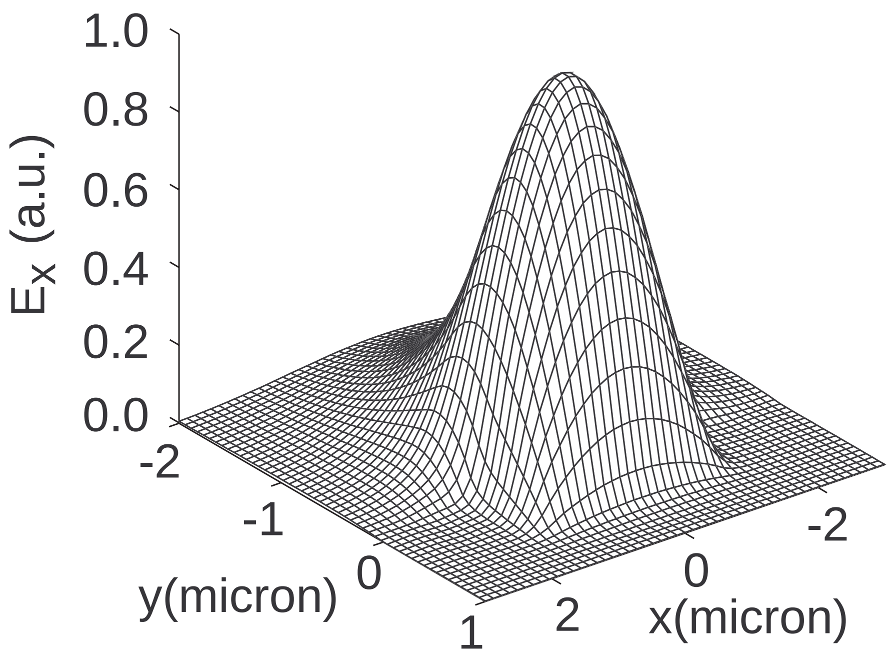
Rib Waveguide: $\mathbf{E}^{\mathbf{x}}_{11}$ mode

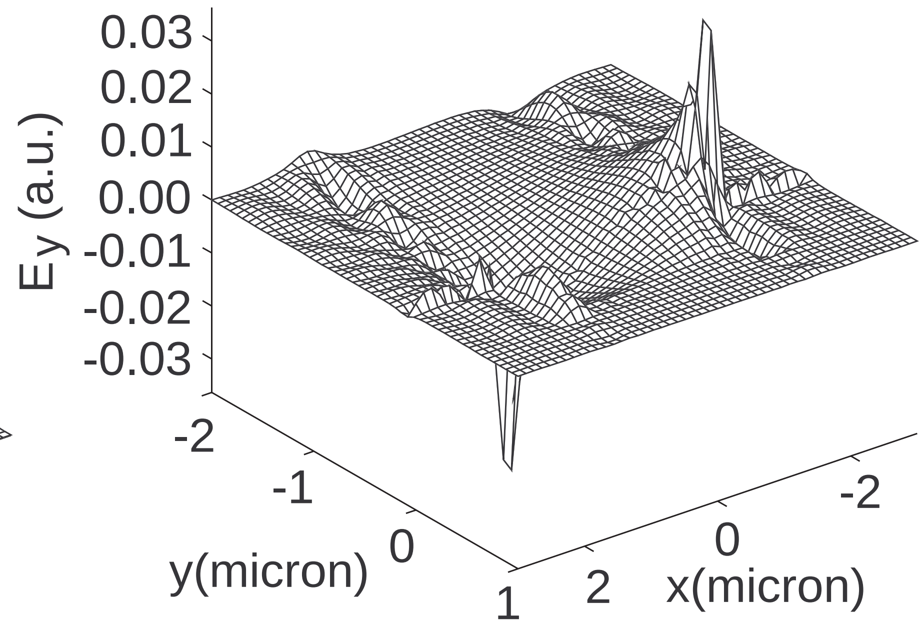
d (μm)	$N_m^e = N_n/2 =$ 10	15	20	25	Full VBPM
0.0	0.2882	0.2949	0.2964	0.2971	0.3011
0.2	0.2962	0.3020	0.3034	0.3040	0.3089
0.4	0.3108	0.3154	0.3165	0.3170	0.3215
0.6	0.3330	0.3363	0.3371	0.3375	0.3415
0.8	0.3643	0.3664	0.3670	0.3672	0.3669
1.0	0.4241	0.4262	0.4268	0.4270	0.4273

Rib Waveguide: $\mathrm{E}^{\mathrm{y}}_{11}$ mode

d	$N_m^o = N_n/2 =$				Full
(μm)	10	15	20	25	VBPM
0.0	0.2495	0.2547	0.2560	0.2566	0.2667
0.2	0.2561	0.2605	0.2614	0.2621	0.2729
0.4	0.2683	0.2716	0.2721	0.2726	0.2839
0.6	0.2879	0.2898	0.2898	0.2902	0.3016
0.8	0.3164	0.3172	0.3169	0.3172	0.3250
1.0	0.3756	0.3777	0.3770	0.3772	0.3854

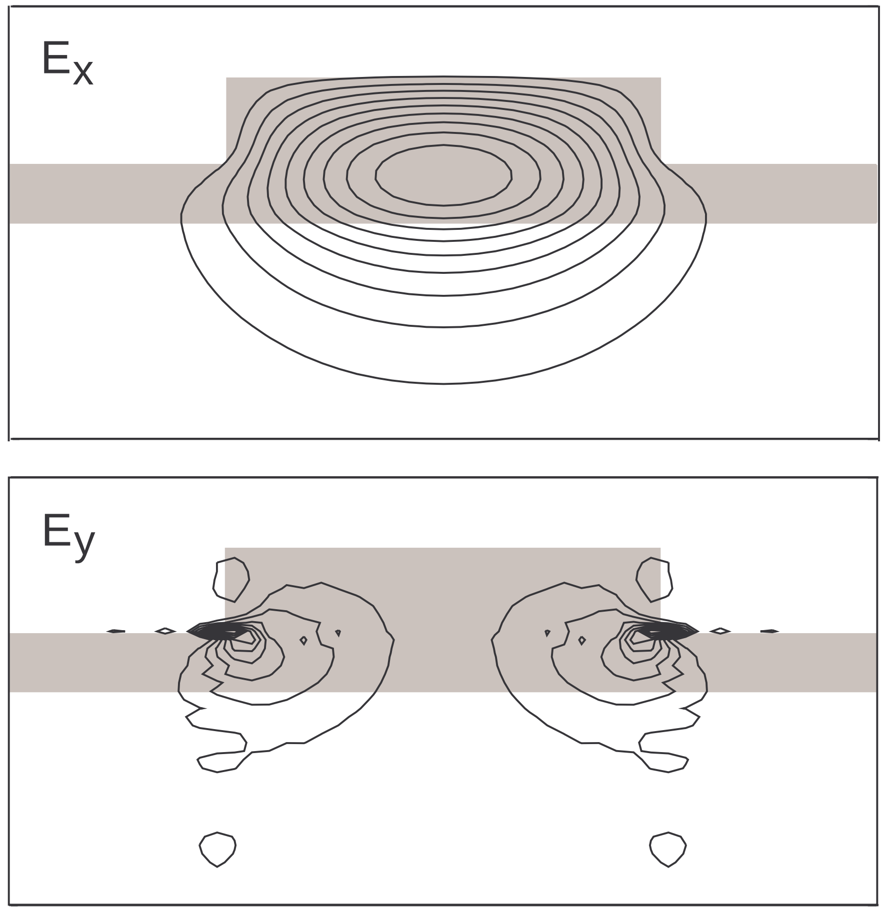
Rib Waveguide: $\mathbf{E}_{11}^{\mathbf{x}}$ mode





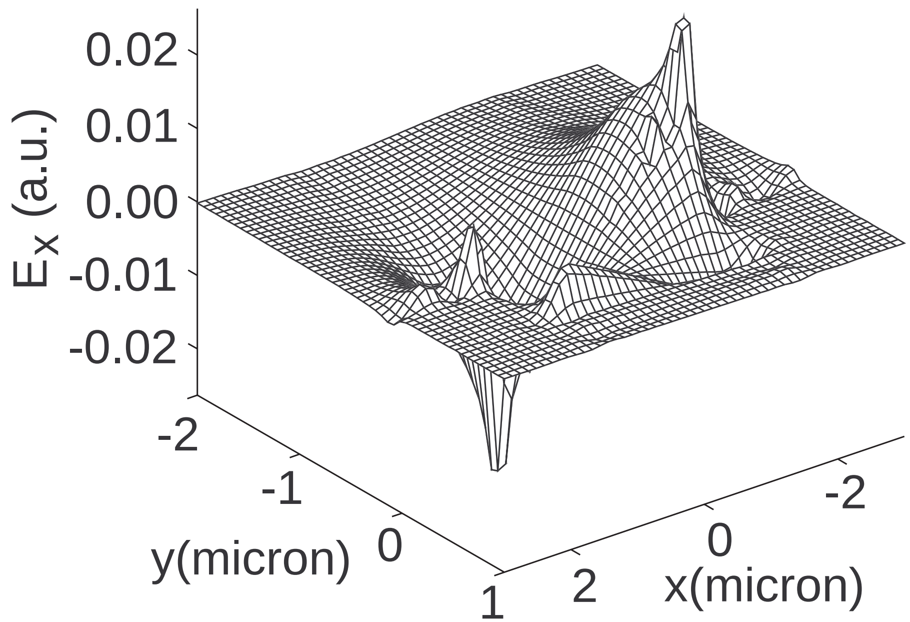
$d = 0.4\mu m$, $N_m^e = N_n/2 = 20$

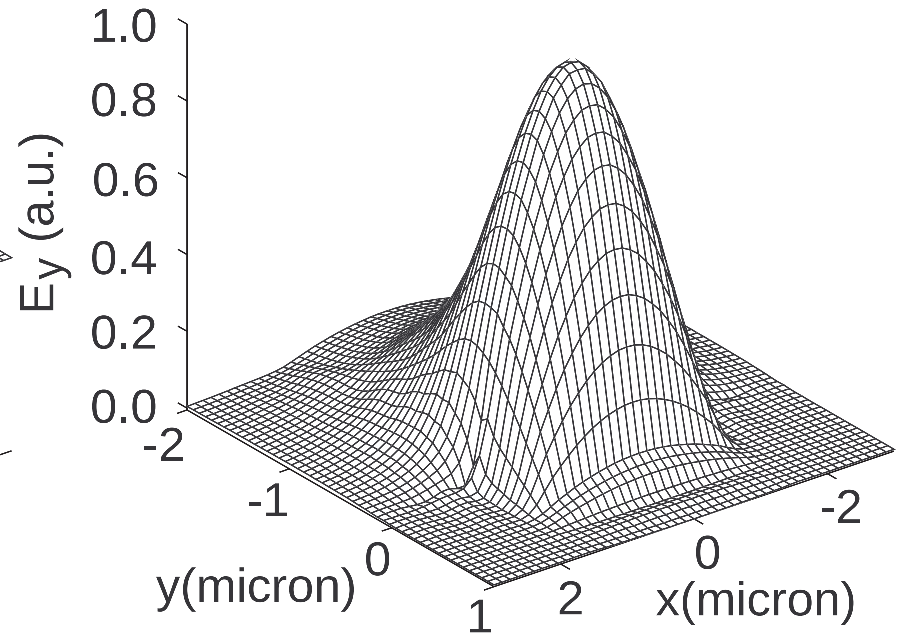
Rib Waveguide: $\mathbf{E}_{11}^{\mathbf{x}}$ mode



$d = 0.4\mu m$, $N_m^e = N_n/2 = 20$

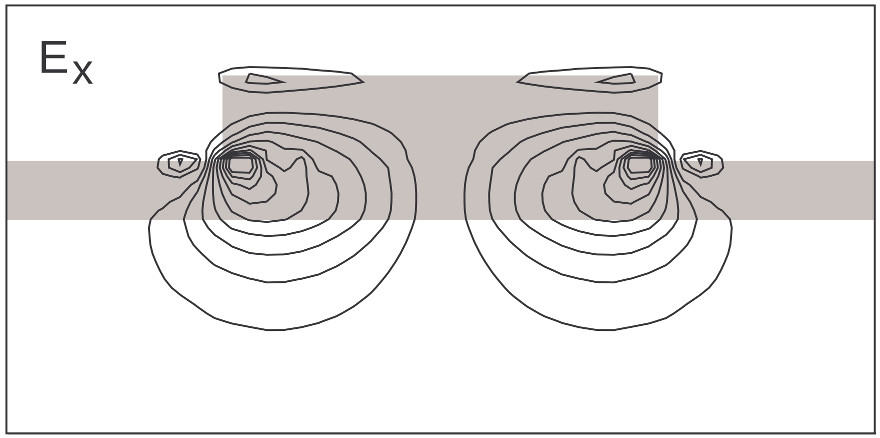
Rib Waveguide: E_{11}^{y} mode

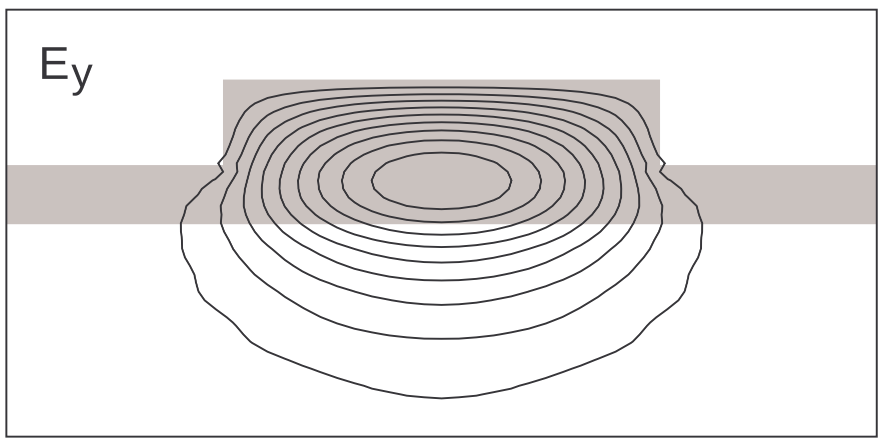




$d = 0.4\mu m$, $N_m^o = N_n/2 = 20$

Rib Waveguide: $\mathbf{E}^{\mathbf{y}}_{11}$ mode





$d = 0.4\mu m$, $N^{o}_{m} = N_n/2 = 20$

Conclusion

- Galerkin's method with mapping for solutions of vector modes of waveguides has been described.
- Specific structures have been studied are rectangular core, circular core and rib waveguide.
- Method's accuracy have been demonstrated by comparison with F-OPT, exact solution and full VBPM.