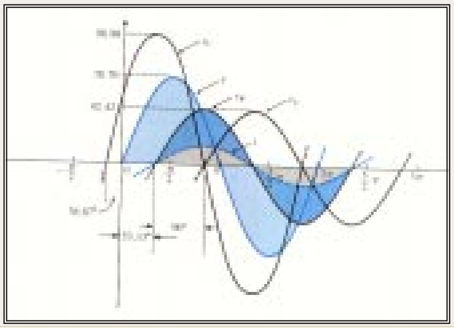
CIRCUIT ANALYSIS II

Chapter 1

Sinusoidal Alternating Waveforms and Phasor Concept

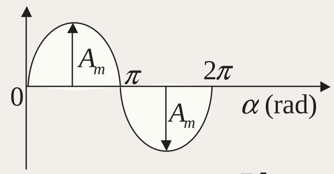


CONTENTS

1.1 Sinusoidal Alternating Waveforms

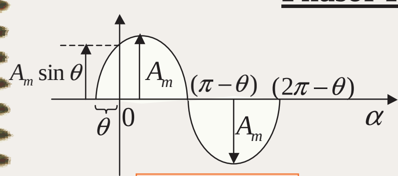
1.1.1 General Format for the Sinusoidal Voltage & Current

$A_m \sin\alpha = A_m \sin\omega t$ Basic mathematical format for the sinusoidal waveform

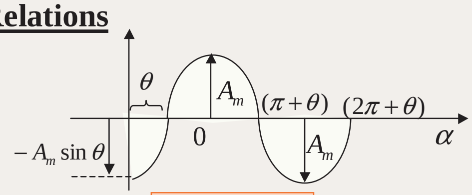


$$i = I_m \sin\omega t = I_m \sin\alpha$$
$$e = E_m \sin\omega t = E_m \sin\alpha$$

Phasor Relations



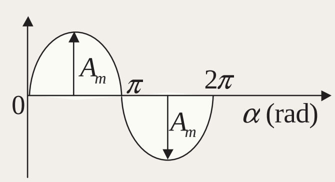
$A_m \sin(\omega t+\theta)$



$A_m \sin(\omega t-\theta)$

1.1.2 Average Value:

$$G \text{ (average value)} = \frac{\text{algebraic sum of the areas}}{\text{length of the curve}}$$

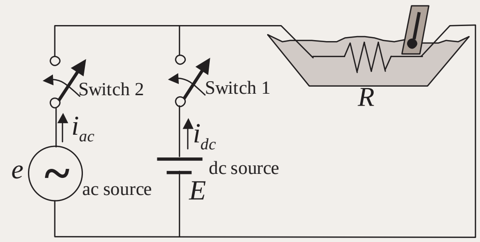


$$G = \frac{\int_0^{2\pi} A_m \sin\alpha}{2\pi} = 0$$

The average value of a pure sinusoidal waveform over one full cycle is zero.

If the waveform equation is: $a = A_0 + A_m \sin\alpha$
then the average value is: $G = A_0$

1.1.3 Effective Values:



The effective value of an ac current is the equivalent dc current that delivers the same power as the ac current.

$$P_{ac} = i_{ac}^2 R = (I_m \sin \omega t)^2 R = (I_m^2 \sin^2 \omega t) R \qquad \sin^2 \omega t = \frac{1}{2}(1 - \cos 2\omega t)$$

$$P_{ac} = \frac{I_m^2 R}{2} - \frac{I_m^2 R}{2} \cos 2\omega t$$

The average power delivered is: $P_{av(ac)} = \dfrac{I_m^2 R}{2}$

Equating the average power delivered by the ac generator to that delivered by the dc source,

$$P_{av(ac)} = P_{dc} \,\Sigma\, \frac{I_m^2 R}{2} = I_{dc}^2 R \,\Sigma\, I_m = \sqrt{2} I_{dc}$$

$$I_{dc} = \frac{1}{\sqrt{2}} I_m = 0.707 \times I_m \quad \Rightarrow \quad I_{eff} = \frac{1}{\sqrt{2}} I_m = 0.707 \times I_m$$

Same thing for the voltage: $E_{eff} = \dfrac{1}{\sqrt{2}} E_m = 0.707 \times E_m$

In general, the effective value of any quantity plotted as a function of time can be found by using the following equation:

$$I_{eff} = \sqrt{\frac{\int_0^T i^2 dt}{T}} = \sqrt{\frac{\text{area}(i^2)}{T}}$$

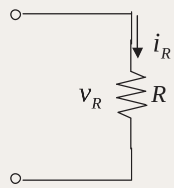
This procedure gives us another designation for the effective value, the ***root-mean-square (rms)*** *value*.

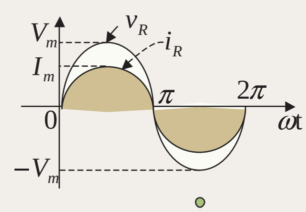
1.2 Basic Elements and Phasors

1.2.1 Elements

Response of the basic R, L and C elements to a sinusoidal voltage and current.

Resistor: $i = \frac{v}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t \Rightarrow \boxed{I_m = \frac{V_m}{R} \Leftrightarrow V_m = I_m R}$





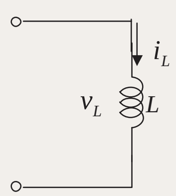
For a purely resistive element, the voltage across and the current through the element are in phase

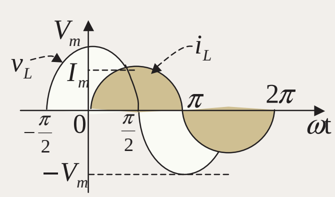
Inductor:

$$v_L = L\frac{di_L}{dt} = L\frac{d}{dt}\left(I_m \sin \omega t\right) = \omega L I_m \cos(\omega t) = V_m \sin\left(\omega t + 90^o\right)$$

$\Rightarrow \boxed{V_m = \omega L I_m}$ $X_L = \omega L$: reactance of the inductor (Ohms).

Inductive reactance is the opposition to the flow of current, which results in the continual interchange of energy between the source and the magnetic field of the inductor. Unlike resistance, reactance does not dissipate electric energy.





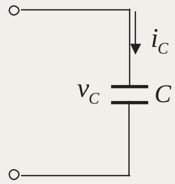
For an inductor v_L leads i_L by 90^o, or i_L lags v_L by 90^o.

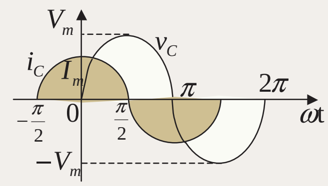
Capacitor:

$$i_C = C\frac{dv_C}{dt} = C\frac{d}{dt}(V_m \sin \omega t) = \omega C V_m \cos(\omega t) = I_m \sin(\omega t + 90^o)$$

$$\boxed{I_m = \omega C V_m \Leftrightarrow V_m = \frac{1}{\omega C} I_m} \qquad X_C = \frac{1}{\omega C} \text{ : reactance of the capacitor (Ohms).}$$

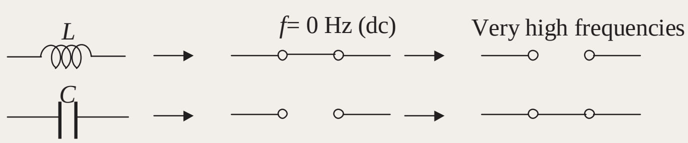
Capacitive reactance is the opposition to the flow of charge, which results in the continual interchange of energy between the source and the electric field of the capacitor. Like the inductor, the capacitor does not dissipate electric energy.





For a capacitor i_C leads v_C by 90^o, or v_C lags i_C by 90^o.

As a general rule for circuit with combined elements: if the current leads the voltage, the circuit is predominantly capacitive, and if the voltage leads the current, it is predominantly inductive.



$$\omega = 2\pi f$$
$$X_L = \omega L$$
$$X_C = \frac{1}{\omega C}$$

Instead of differentiation, it is possible to use integration depending on the unknown quantities:

$$i_L = \frac{1}{L}\int v_L dt \qquad v_C = \frac{1}{C}\int i_C dt$$

1.2.2 Average Power and Power Factor

In general the instantaneous power is determined by: $p = vi$

In general, for a sinusoidal voltage and current, the average power is determined by: $P = \frac{V_m I_m}{2}\cos\theta = V_{eff} I_{eff}\cos\theta$

Where θ is the phase angle between the voltage and the current.

$V_{eff} = V_{rms} = \frac{V_m}{\sqrt{2}}$ Effective voltage

$I_{eff} = I_{rms} = \frac{I_m}{\sqrt{2}}$ Effective current

$PF = \cos\theta$ Power Factor

Resistor	***Inductor***	***Capacitor***
$P = \frac{V_m I_m}{2}\cos 0 = V_{eff} I_{eff}$	$P = \frac{V_m I_m}{2}\cos 90^o = 0$	$P = \frac{V_m I_m}{2}\cos 90^o = 0$

1.2.3 Rectangular and Polar Representation

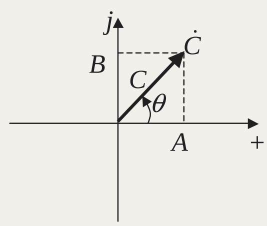
$\dot{C} = A + jB$ Rectangular form

$\dot{C} = C\langle\theta$ Polar form

Polar to rectangular

$A = C\cos\theta$

$B = C\sin\theta$



Rectangular to polar

$C = \sqrt{A^2 + B^2}, \theta = \tan^{-1}\frac{B}{A}$

Example:

$v_1 = V_{m1}\sin(\omega t + \theta_1) \Sigma\ V_{m1}\langle\theta_1$

$v_2 = V_{m2}\sin(\omega t + \theta_{21}) \Sigma\ V_{m2}\langle\theta_2$

$v = v_1 + v_2 = V_{m1}\langle\theta_1 + V_{m2}\langle\theta_2$

Use the calculator directly with polar form or convert v_1 and v_2 to rectangular form, do the summation and convert the result to polar form.

1.2.4 Impedance and Phasor Diagram

Resistor: $\dot{V} = \dot{I}\dot{Z}_R \; \Sigma \; \dot{Z}_R = R\langle 0^o = R + j0$

$I_m = \frac{V_m}{R}$ or $V_m = I_m R$ In phasor form $\dot{V} = V\langle 0^o \; \Sigma \; \dot{I} = \frac{V}{R}\langle 0^o = I\langle 0^o$

Inductor: $\dot{V} = \dot{I}\dot{Z}_L \; \Sigma \; \dot{Z}_L = X_L\langle +90^o = \omega L\langle +90^o = 0 + jX_L$

$I_m = \frac{V_m}{\omega L}$ or $V_m = I_m X_L$

In phasor form $\dot{V} = V\langle 0^o \; \Sigma \; \dot{I} = \frac{V}{X_L}\langle -90^o = I\langle -90^o$

Capacitor: $\dot{V} = \dot{I}\dot{Z}_C \; \Sigma \; \dot{Z}_C = X_c\langle -90^o = \frac{1}{\omega C}\langle -90^o = 0 - jX_C$

$I_m = \omega C V_m$ or $V_m = I_m X_C$

In phasor form $\dot{V} = V\langle 0^o \; \Sigma \; \dot{I} = \frac{V}{X_C}\langle +90^o = I\langle +90^o$

1.2.5 Kirchhoff's Laws

- ***Voltage Law:*** *the complex (vector) sum of the potential rises and drops around a closed loop (or path) is zero.*

$$\sum_{O} V = 0 \Leftrightarrow \sum_{O} V_{rise} = \sum_{O} V_{drops}$$

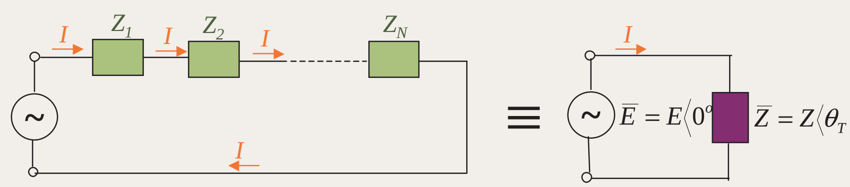
- ***Current Law:*** *the complex (vector) sum of the currents entering and leaving a junction is zero. Or the sum of currents entering a junction must equal the sum of currents leaving the junction.*

$$\sum I_{entering} = \sum I_{leaving}$$

1.3 Series and Parallel Circuits

1.3.1 Series Circuits

Two elements are in series if they have only one point of intersection that is not connected to other current-carrying elements of the network.



For N impedances in series the equivalent impedance is:

$$\bar{Z}_T = \bar{Z}_1 + \bar{Z}_2 + \cdots + \bar{Z}_N = Z_T \angle \theta_T$$

Impedance amplitude

Impedance angle

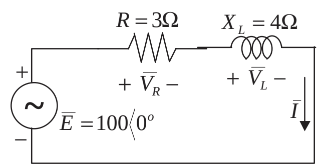
Current in the circuit: $\bar{I} = \dfrac{\bar{E}}{\bar{Z}_T} = \dfrac{E\angle 0^o}{Z_T\angle\theta_T} = I\angle -\theta_T$ (A)

Voltage across each element: $\bar{V}_1 = \bar{I}\bar{Z}_1,\ \ \bar{V}_2 = \bar{I}\bar{Z}_2,\ \ \cdots,\ \ \bar{V}_N = \bar{I}\bar{Z}_N$ (V)

Power delivered by each impedance : $P_i = I^2 R_i$ (W) $(\mathrm{i} = 1, \cdots, \mathrm{N})$

Power delivered by the source: $P_T = EI\cos\theta_T = P_1 + P_2 + \cdots + P_N$ (W)

Example of Series R-L Circuit:



Series R-L Circuit

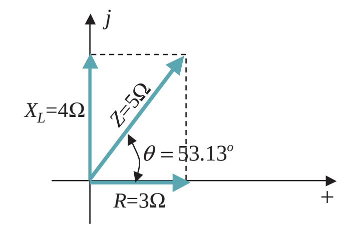
$$\bar{Z}_T = \bar{Z}_1 + \bar{Z}_2 = 3\angle 0^o + 4\angle 90^o = 3 + j4$$

Rectangular form

$$\bar{Z}_T = 5\angle 53.13^o$$

Polar form

Impedance diagram:



Current: $\bar{I} = \dfrac{\bar{E}}{\bar{Z}_T} = \dfrac{100\angle 0^o}{5\angle 53.13^o} = 20\angle -53.13^o \text{ (A)}$

Voltages:

Polar form:

$$\bar{V}_R = \bar{I}\bar{Z}_1 = 20\angle -53.13^o \times 3\angle 0^o = 60\angle -53.13^o \text{ (V)}$$

$$\bar{V}_L = \bar{I}\bar{Z}_2 = 20\angle -53.13^o \times 4\angle 90^o = 80\angle +36.87^o \text{ (V)}$$

Rectangular form:

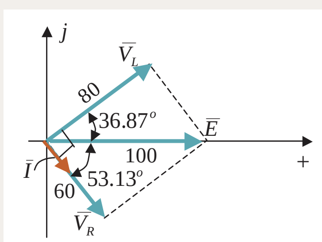
$$\bar{V}_R = 36 - j48 \text{ (V)}$$

$$\bar{V}_L = 64 + j48 \text{ (V)}$$

Apply *Kirchhoff's voltage law:* $\sum_O \bar{V} = \bar{E} - \bar{V}_R - \bar{V}_L = 0 \ \Sigma \ \bar{E} = \bar{V}_R + \bar{V}_L$

$$\bar{E} = (36 - j48) + (64 + j48) = 100 + j0 = 100\angle 0^o$$

Phasor diagram:



Power:

Method 1: $P_T = EI\cos\theta_T = 100\times 20\times\cos(53.13^o) = 1200\ \text{W}$

Method 2: $P_T = I^2R = 20^2\times 3 = 1200\ \text{W}$

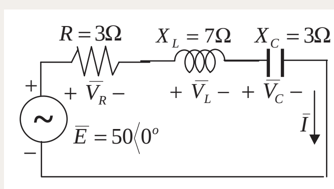
Method 3:
$$P_T = EI\cos\theta_T = P_R + P_L = V_R I\cos\theta_R + V_L I\cos\theta_L$$
$$= 60\times 20\times\cos 0^o + 80\times 20\times\cos 90^o$$
$$= 1200 + 0 = 1200\ \text{W}$$

Power Factor: $PF = \dfrac{P_T}{EI} = \cos\theta = \cos -53.13^o = 0.6\ \text{lagging}$

Impedance angle = angle between input voltage and current

When current is lagging the voltage, the power factor is said to be a: *lagging power factor*.

Example of Series R-L-C Circuit:



Series R-L-C Circuit

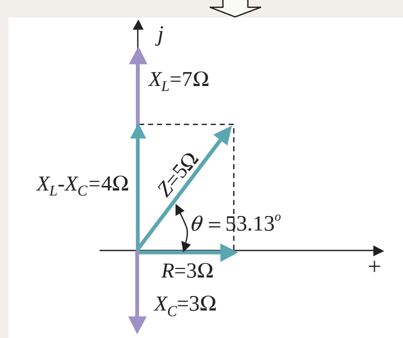
$$\bar{Z}_T = \bar{Z}_1 + \bar{Z}_2 + \bar{Z}_3 = R\langle 0^o + X_L\langle 90^o + X_C\langle -90^o$$
$$= 3 + j7 - j3 = 3 + j4$$

Rectangular form

$$\bar{Z}_T = 5\langle 53.13^o$$

Polar form

Impedance diagram:



Current: $\bar{I} = \dfrac{\bar{E}}{\bar{Z}_T} = \dfrac{50\angle 0^o}{5\angle 53.13^o} = 10\angle -53.13^o \ \text{(A)}$

Voltages:

Polar form:

$$\bar{V}_R = \bar{I}\bar{Z}_1 = 10\angle -53.13^o \times 3\angle 0^o = 30\angle -53.13^o \ \text{(V)}$$
$$\bar{V}_L = \bar{I}\bar{Z}_2 = 10\angle -53.13^o \times 7\angle 90^o = 70\angle +36.87^o \ \text{(V)}$$
$$\bar{V}_C = \bar{I}\bar{Z}_3 = 10\angle -53.13^o \times 3\angle -90^o = 30\angle -143.13^o \ \text{(V)}$$

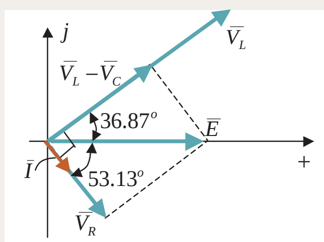
Rectangular form:

$$\bar{V}_R = 18 - j24 \ \text{(V)}$$
$$\bar{V}_L = 56 + j42 \ \text{(V)}$$
$$\bar{V}_C = -24 - j18 \ \text{(V)}$$

Apply *Kirchhoff's voltage law:*

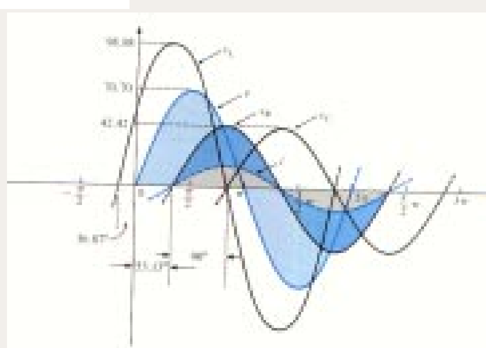
$$\sum\nolimits_{O} \bar{V} = \bar{E} - \bar{V}_R - \bar{V}_L - \bar{V}_C = 0 \ \sum \ \bar{E} = \bar{V}_R + \bar{V}_L + \bar{V}_C$$
$$\bar{E} = (18 - j24) + (56 + j42) + (-24 - j18) = 50 + j0 = 50\angle 0^o$$

Phasor diagram:



Time domain:

$$i = \sqrt{2} \times 10 \sin(\omega t - 53.13^o)$$
$$v_R = \sqrt{2} \times 30 \sin(\omega t - 53.13^o)$$
$$v_L = \sqrt{2} \times 70 \sin(\omega t + 36.87^o)$$
$$v_C = \sqrt{2} \times 30 \sin(\omega t - 143.13^o)$$



Power:

Method 1: $P_T = EI\cos\theta_T = 50\times10\times\cos(53.13^o) = 300\ \text{W}$

Method 2: $P_T = I^2R = 10^2\times3 = 300\ \text{W}$

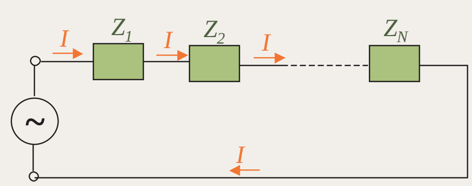
Method 3:
$$P_T = EI\cos\theta_T = P_R + P_L + P_C = V_RI\cos\theta_R + V_LI\cos\theta_L + V_CI\cos\theta_C$$
$$= 30\times10\times\cos 0^o + 70\times10\times\cos 90^o + 30\times10\times\cos 90^o$$
$$= 300 + 0 + 0 = 300\ \text{W}$$

Power Factor:

$$PF = \frac{P_T}{EI} = \cos\theta = \frac{R}{Z_T} = 0.6\ \text{lagging}$$

Impedance angle = angle between input voltage and current

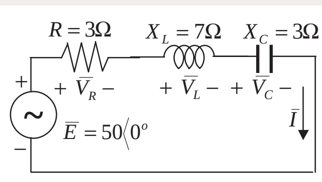
Voltage Divider Rule:



$$I = \frac{E}{Z_T} = \frac{V_i}{Z_i}$$

$$V_i = \frac{Z_iE}{Z_T}$$

Example:



Series R-L-C Circuit

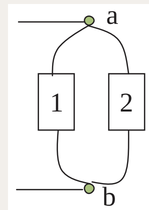
$$V_R = \frac{Z_RE}{Z_T} = \frac{RE}{R + jX_L - jX_C} = \frac{3\times50}{3 + j(7-3)} = \frac{150}{5\angle53.13^o} = 30\angle-53.13^o$$

$$V_L = \frac{Z_LE}{Z_T} = \frac{jX_LE}{R + jX_L - jX_C} = \frac{7\angle90^o\times50}{3 + j(7-3)} = \frac{350\angle90^o}{5\angle53.13^o} = 70\angle36.87^o$$

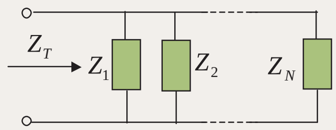
$$V_C = \frac{Z_CE}{Z_T} = \frac{-jX_CE}{R + jX_L - jX_C} = \frac{3\angle-90^o\times50}{3 + j(7-3)} = \frac{150\angle90^o}{5\angle53.13^o} = 30\angle-143.13^o$$

1.3.2 Parallel Circuits

Two elements, branches, or networks are in parallel if they have two points in common.



Parallel Impedances:



$$Y_i = \frac{1}{Z_i} = \text{Admittance}$$

$$\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N} \Rightarrow Y_T = Y_1 + Y_2 + \cdots + Y_N$$

Total impedance of parallel impedances is always less than the value of the smallest impedance

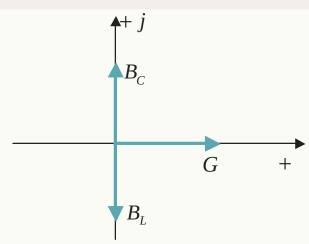
For two parallel impedances: $Z_T = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$

For three parallel impedances: $Z_T = \dfrac{Z_1 Z_2 Z_3}{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}$

Conductance: $G = \dfrac{1}{R}$ (siemens, S)

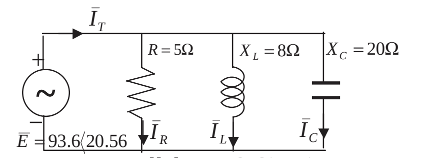
Susceptance: $B_L = \dfrac{1}{X_L}$ (siemens, S)

$B_C = \dfrac{1}{X_C}$ (siemens, S)



Admittance Diagram

Example of Parallel R-L-C Circuit:



Parallel R-L-C Circuit

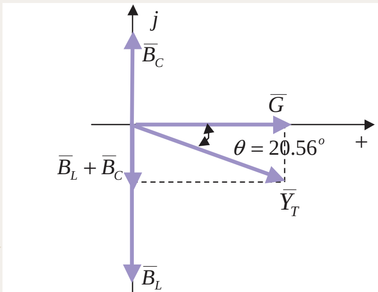
$$\bar{Y}_1 = \bar{G} = G\angle 0^o = \frac{1}{R}\angle 0^o = 0.2\angle 0^o$$

$$\bar{Y}_2 = \bar{B}_L = B_L\angle -90^o = \frac{1}{X_L}\angle -90^o = 0.125\angle -90^o$$

$$\bar{Y}_3 = \bar{B}_C = B_C\angle 90^o = \frac{1}{X_C}\angle 90^o = 0.05\angle 90^o$$

$$\bar{Y}_T = \bar{Y}_1 + \bar{Y}_2 + \bar{Y}_3 = (0.2 + J0) + (0 - J0.125) + (0 + J0.05)$$
$$= 0.2\text{-}j0.075 = 0.2136\angle -20.56^o$$

$$\bar{Z}_T = \frac{1}{\bar{Y}_T} = \frac{1}{0.2136\angle -20.56^o} = 4.68\angle 20.56^o$$



Admittance Diagram

Current:

$$\bar{I}_T = \frac{\bar{E}}{\bar{Z}_T} = \frac{93.6\angle 20.56^o}{4.68\angle 20.56^o} = 20\angle 0^o$$

$$\bar{I}_R = \bar{E}\bar{G} = 93.6\angle 20.56^o \times 0.2\angle 0^o = 18.72\angle 20.56^o$$

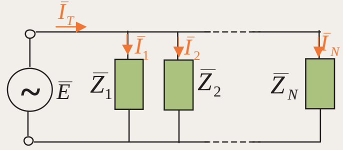
$$\bar{I}_L = \bar{E}\bar{B}_L = 93.6\angle 20.56^o \times 0.125\angle -90^o = 11.7\angle 69.44^o$$

$$\bar{I}_C = \bar{E}\bar{B}_C = 93.6\angle 20.56^o \times 0.05\angle 90^o = 4.68\angle 110.56^o$$

Apply *Kirchhoff's current law:*

$$\bar{I}_T = \bar{I}_R + \bar{I}_L + \bar{I}_C \;\Sigma\; \bar{I}_T - \bar{I}_R - \bar{I}_L - \bar{I}_C = 0$$
$$(20 + j0) - (17.527 + j6.574) - (4.109 - 10.955) - (-1.643 + j4.382) \cong 0 + j0$$

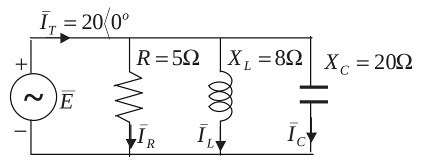
Current Divider Rule:



$$\bar{E} = \bar{I}_T \bar{Z}_T = \bar{I}_i \bar{Z}_i$$

$$\bar{I}_i = \frac{\bar{Z}_T \bar{I}_T}{\bar{Z}_i}$$

Example:



Parallel R-L-C Circuit

$$\bar{Z}_T = 4.68\angle 20.56^o$$

$$\bar{I}_R = \frac{\bar{Z}_T \bar{I}_T}{\bar{Z}_R} = \frac{4.68\angle 20.56^o \times 20\angle 0^o}{5\angle 0^o} = 18.72\angle 20.56^o$$

$$\bar{I}_L = \frac{\bar{Z}_T \bar{I}_T}{\bar{Z}_L} = \frac{4.68\angle 20.56^o \times 20\angle 0^o}{8\angle 90^o} = 11.7\angle -69.44^o$$

$$\bar{I}_C = \frac{\bar{Z}_T \bar{I}_T}{\bar{Z}_C} = \frac{4.68\angle 20.56^o \times 20\angle 0^o}{20\angle -90^o} = 4.68\angle 110.56^o$$

Summary:

Kirchhoff's law applies to ac circuits the same way it applies to dc circuits.

Series and parallel circuit rules apply to ac circuits the same way they apply to dc circuits. Instead of resistance R impedance Z is used.

Series — impedance

$$Z_T = \sum_{i=1}^{N} Z_i$$

Parallel — admittance

$$\bar{Y}_T = \frac{1}{\bar{Z}_T} = \sum_{i=1}^{N} \frac{1}{\bar{Z}_i}$$

For any network configuration(series, parallel or series-parallel), the angle θ_T by which the applied voltage leads the source current will be positive for inductive network and negative for capacitive networks.