Part 1: Al-Khwārizmī, Quadratic Equations, and the Birth of Algebra

1 Introduction

We look at *Kitāb al-muḫtaṣar fī ḥisāb al-jabr wa al-muqābala,* arguably the first book on algebra. It was written around 825, probably in Baghdad, by abū Ja'far Muḥammad ibn Mūsā al-Khwārizmī. He mainly worked on astronomical tables, and of course astrology.

The word *jabr* is Arabic, and means "putting together." The *jabr* in the title refers to taking subtracted terms to the "other side," as in: if $5x - 7 = 40 - x$ then $6x = 47$. But that's not what the book is about.

I can't find the word *jabr* in the main text, and found *muqābala* only once. So the words must have had a mathematical meaning before al-Khwārizmī. It has been argued that ibn Turk wrote a book on *al-jabr* before al-Khwārizmī.[1]

Al-Khwārizmī is said to have written a book on what is usually called the Indo-Arabic number system, the now almost universal way of representing numbers. This work (maybe) was translated into Latin in the twelfth century; it is not extant in Arabic.

In Western Europe, the newly introduced methods of representation and computation somehow got associated with al-Khwārizmī. For several centuries thereafter these arithmetical techniques were called *algorismus.*[2] But in his algebra book al-Khwārizmī writes out all numbers in words!

Al-Khwārizmī's *al-jabr wa al-muqābala* was hugely influential. Several books with similar titles appeared within the next 100 years, notably those of ibn Turk, Thābit ibn Qurra, abū Kāmil, and ibn al-Fatḥ. They borrow problems shamelessly from al-Khwārizmī.

[1] Ultimately *al-jabr* mutated into *algebra.* In Spanish, *algebrista* once meant bone-setter or surgeon, and is still sometimes used in that sense. In Portuguese, *algebrar* means to deal with bone fractures or dislocations. Related words had a medical meaning in late Middle English—look at the complete Oxford dictionary.

[2] There were many variants—Chaucer used *augrim.* And a decimal digit, or string of digits, is still called *guarismo* in Spanish.

There are twelfth-century Latin translations by Gerard of Cremona, Robert of Chester and maybe others, but the oldest surviving copy is from the fourteenth century. The work also reached Europe through Hebrew adaptations. And the mathematician Leonardo of Pisa ("Fibonacci") had direct contact with North Africa and the Near East. Many problems in his *Liber abbaci* (1202) are taken without attribution from abū Kāmil. The only acknowledgment is the word "Maumet" in the margin at the beginning of his treatment of algebra, presumably a reference to al-Khwārizmī.

2 A Sample from *al-jabr wa al-muqābala*

The following extract is from Frederick Rosen's 1831 translation of one of the three Arabic manuscripts known to have survived. Of course the manuscript is a copy of a copy of a copy ..., and copies were seldom faithful before the age of Xerox. Translation brings about its own inevitable distortions. But I compared the extract with the two known Latin translations, and though there are large differences in wording, the mathematical contents are substantially the same.

> *Squares and Numbers are Equal to Roots;* for instance, "a square and twenty-one in numbers are equal to ten roots of the same square." That is to say, what must be the amount of a square, which, when twenty-one dirhems are added to it, becomes equal to the equivalent of ten roots of that square? Solution: Halve the number of the roots; the moiety is five. Multiply this by itself; the product is twenty-five. Subtract from this the twenty-one which are connected with the square; the remainder is four. Extract its root; it is two. Subtract this from the moiety of the roots, which is five; the remainder is three. This is the root of the square which you required, and the square is nine. Or you may add the root to the moiety of the roots; the sum is seven; this is the root of the square which you sought for, and the square itself is forty-nine.
>
> When you meet with an instance which refers you to this case, try its solution by addition, and if that do not serve,[3] then subtraction certainly will. For in this case both addition and subtraction may be employed, which will not answer in any other

[3] There is a built-in assumption here which lasted for centuries, namely that a problem has a single answer. So what is being said here is *not* that this type of equation may have two solutions, but that if addition doesn't give the answer, then subtraction will.

> of the three cases in which the number of roots must be halved. And know that, in a question belonging to this case you have halved the number of the roots and multiplied the moiety by itself, if the product be less than the number of dirhems connected with the square, then the instance is impossible; but if the product be equal to the dirhems by themselves, then the root of the square is equal to the moiety of the roots alone, without addition or subtraction.
>
> In every instance where you have two squares, or more or less, reduce them to one entire square, as I have explained under the first case.

The words "square" and "root" don't quite convey the flavour of the original. Where Rosen writes "square," the Arabic reads *māl.* The literal meaning of *māl* is wealth. Early Islamic algebraists are often looking for x^2, not x. The latter is called *jadhr*, which literally means root in the botanical sense. The word *shaī* is also often used for an unknown quantity. The ordinary meaning of *shaī* is thing. And maybe al-Khwārizmī was uncomfortable with pure numbers. The phrase "twenty-one dirhems" means 21 units, or just plain 21. (The dirham is a unit of currency.)

There is reason to think that the part "if the product be less . . . is impossible" is a later interpolation or reinterpretation—al-Khwārizmī may have been unclear about the role of the sign of the discriminant.

3 Al-Khwārizmī's Six Types

Remember in what follows that all numbers are positive. The problem discussed in Section 2 is of the type "squares and numbers equal to roots." In modern notation, this is the class of equations of the shape

$$ax^2 + c = bx,$$

where a, b, and c are fixed positive but otherwise arbitrary quantities. There was no available way then to talk about general parameters[4] a, b, and c. Thus specific numbers had to be used. That had gone on for many centuries, and only ended around 1600 with the work of Viète. (The preceding comment is made often. It is an oversimplification—there are isolated instances of parameter use before Viète.)

[4]This is not strictly true. Euclid's letter labels may be thought of as parameters. For a long time, geometry was viewed as the only "general" mathematical science.

Al-Khwārizmī pretends to be looking at the particular equation $x^2+21 = 10x$, but his description of the *process* of solving equations of this type is quite general. The following are al-Khwārizmī's six types.

1. squares equal to roots ($ax^2 = bx$)
2. squares equal to number ($ax^2 = c$)
3. roots equal to number ($ax = c$)
4. squares and roots equal to number ($ax^2 + bx = c$)
5. squares and number equal to roots ($ax^2 + c = bx$)
6. roots and number equal to squares ($bx + c = ax^2$)

The first three types are easy. Al-Khwārizmī gives algorithms to express the (positive) solutions of the last three. He doesn't have parameters, only specific numbers, so he can't supply formulas in the current sense of the word. But any careful reader would know what to do. Note that for all types except type 5, there is a unique solution, but as al-Khwārizmī observes (see Section 2), equations of type 5 have two, none, or one.

4 Al-Khwārizmī's Geometric Arguments

After introducing the six types and giving in great detail rules for solving sample equations of each type, al-Khwārizmī proceeds to give geometric justifications for each rule, again ostensibly only in particular numerical cases. The geometry used is informal, as opposed to the formal geometry of Euclid.[5]

For "square and ten roots equal to thirty-nine" (type 4) al-Khwārizmī gives two different arguments, based on pictures somewhat like the ones below. Such pictures would be endlessly reproduced in later algebra books.

Look first at the diagram on the left of Figure 1. Imagine drawing a square (the inner square) whose side is the unknown number x. On each side of this square, make a rectangle whose other side has length 10/4. The inner square together with its four wings has area $x^2 + 10x$, which is 39. "Complete" the picture by adding the four dashed squares at the corners.

[5] Al-Khwārizmī was probably well-acquainted with Euclid's *Elements* but chose not to quote it for good pedagogical reasons.

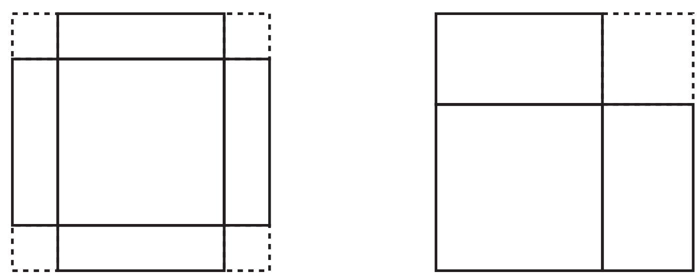

Figure 1: The Equation $x^2 + 10x = 39$

These have sides $10/4$, so their combined area is 25. It follows that the "outer" square has area $39 + 25$, and therefore side 8. To find the side of the inner square, subtract $2(10/4)$ from 8.

The original diagrams are elaborately labelled. It takes al-Khwārizmī more than a page to run through the argument. Note that the scale of Figure 1 is grossly off: the inner square turns out to be 3×3 while the corner squares are 2.5×2.5. In the original diagrams, the scale is also quite wrong. That may have been deliberate, to suggest generality.

Al-Khwārizmī presents a second argument, using a picture like the diagram on the right of Figure 1. The bottom left square has side which is the unknown x. On two adjacent sides of it make a rectangle whose other side is $10/2$. Complete this picture by adding the dashed 5×5 square at the upper right, and the rest is easy. The phrase "completing the square" probably comes from these pictures.

The picture for "square and twenty-one equal to ten roots" looks more complicated. Draw a square $abcd$ whose area represents the unknown x^2. Add a rectangular extension $aefb$ with area 21. Since $x^2 + 21 = 10x$, it

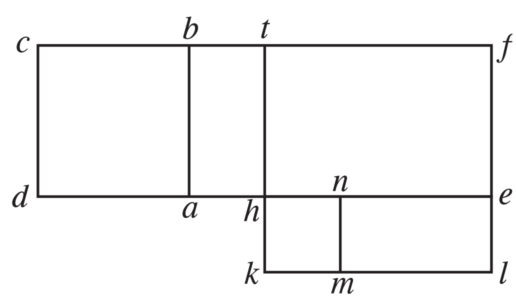

Figure 2: The Equation $x^2 + 21 = 10x$

follows that side cf has length 10. Let t be the midpoint of side cf, and construct the square $tklf$. Let tk meet the line de at h, and construct the square $hkmn$.

It is easy to see that $ah = hk$ and therefore $ab = ne$. Since the area of rectangle $aefb$ is 21, so is the area of $thef$ together with the area of $mlen$. But the area of rectangle $tklf$ is 25, so square $hkmn$ has area 4 and therefore side 2. It follows that th has side $5 - 2$, and therefore $x = 3$.

The above is a condensed version of al-Khwārizmī's argument. It is not the most elegant possible; you might want to improve on it. And what about the root $x = 7$? You can have the pleasure of drawing an appropriate diagram, and also of producing a geometric justification for an equation like "three roots plus four equal to square."

5 Rough Outline of *al-jabr wa al-muqābala*

The book can be conceptually divided into six parts.

1. **Classification of equations and first worked examples**

2. **Geometric justification of the procedures**

3. **Some basic algebraic manipulations** This contains a detailed description of how to expand products such as $(x+a)(x+b)$, $(x+a)(x-b)$, and $(x-a)(x-b)$. Of course there are no symbols, everything is written out in words, and a and b are always specific numbers. It all takes quite a while. Then there are some observations on the manipulation of surds, for instance $3\sqrt{x} = \sqrt{3^2 \cdot x}$.

4. **Worked problems** There are 39 of them. Each problem is first reduced to a linear or quadratic equation. Al-Khwārizmī doesn't bother to work out the rest of the calculation in detail. He just refers to the methods he has described earlier, or just writes down the answer. Here is a sample problem:

   > I have divided ten into two parts, and have divided the first by the second, and the second by the first, and the sum of the results is two and one-sixth.

5. **Applications to Commerce and Geometry** The commercial part is mercifully short (sample: "ten for six, how many for four?"), and there are some thousand year old low-level rules for calculating areas and volumes. The rest is conceptually more interesting. Al-Khwārizmī looks

at some geometric problems and shows how to reduce them to solving a quadratic equation. In this way he finds, for example, the dimensions of the square inscribed in a triangle that has sides 10, 10, and 12.

6. **Problems of inheritance** Islamic inheritance law is very complex. Al-Khwārizmī shows how to use algebra to solve a large number of problems about legacies. The equations involved are all linear, but setting them up requires taking into account subtle points of law.[6] Inheritance questions take up roughly the last half of al-Khwārizmī's book; they were left out in Latin translations.

6 Evaluating *al-jabr wa al-muqābala*

6.1 The positive

Many writers have pointed to what they call quadratic equations, and their solution, well before the time of al-Khwārizmī. They have "seen" quadratic equations in old Babylonian documents (around -1700), in late Babylonian work (around -300), in the work of Heron of Alexandria (50?), Diophantus of Alexandria (250?), Āryabhata (500), Brahmagupta (650), and others.

In the last twenty years, this view has been vigorously challenged by the historian Rushdī Rāshid, who argues that al-Khwārizmī's algebra represents a radical departure from the past. Views of who did what first have been often coloured by ethnic or national ties, but I think that Rāshid is fundamentally right.

Certainly there are many problems that we would *now* probably solve by using the quadratic formula, and that were solved sporadically from old Babylonian times on by systematic methods. But al-Khwārizmī may have been the first person to discover the *notion* of quadratic equation, and to begin the development of a *theory* of equations.

This is a delicate point, so I go into some detail. Consider for example the problem "find two numbers whose sum is 12 and whose product is 35." A standard school approach goes as follows. Let x be one of the numbers. Then the other is $12 - x$, and therefore $x(12 - x) = 35$. Rewrite this as $x^2 - 12x + 35 = 0$ and use the quadratic formula.

In more or less the same way, but without symbols, al-Khwārizmī would arrive at "square and thirty-five equals twelve roots," then use his method for solving equations of type 4.

[6] In medieval times, algebra was compulsory for students of law at the University of Cairo.

Here are a couple of older approaches that do not involve setting down a three-term quadratic equation. At the risk of creating confusion, I will use modern notation.

Let one of the numbers be $6+t$. Then the other is $6-t$, and the product is $36-t^2$. It follows that $36-t^2=35$, so $t^2=1$. This kind of approach is frequently found in Diophantus. It has Babylonian roots.

Or else let the numbers be x and y, with $x>y$. Consider the identity

$$(x+y)^2=(x-y)^2+4xy.$$

Here $x+y=12$ and $xy=35$, so $(x-y)^2=144-140=4$, and therefore $x-y=2$. Thus $x=(x+y)/2+(x-y)/2=7$. This approach, without explicit justification, also goes back to Babylonian times, and was also used in China.

I like these two approaches much better than the "quadratic equation" approach, since they preserve symmetry. But the idea of producing a theory of equations, and of systematically reducing problems to the solution of equations would prove to be tremendously fruitful in the future. And there is good reason to think that this approach was initiated by al-Khwārizmī—his near contemporaries were definite on this point.

6.2 The Negative

Even if we take into account the likelihood that al-Khwārizmī intended to write a "popular" book and not a scholarly one, the fact remains that the book is technically quite weak. Al-Khwārizmī is simply not in the same league as the Greek masters who preceded him, or the good Islamic algebraists who came after him. His geometric justification for equations of type 5 and 6 is clumsy, and for type 5 it is incomplete.

A case can be made that Diophantus is the first person to do algebra. To some degree, he uses symbolic notation, while al-Khwārizmī does not. And he is technically far stronger than al-Khwārizmī. There is also a possibility that a theory of equations was known in China before al-Khwārizmī, or that he got his theory of equations from an unknown late Babylonian source, or an Indian source.

Well before al-Khwārizmī's time, India had a strong mathematical tradition, and some writers, in particular Indian ones, have argued for an Indian genesis of algebra. The Indian literature indeed has many problems that we probably would attack by reducing to a quadratic equation. But the documents that survive don't make the theoretical framework explicit—most

Indian mathematicians wrote in verse! We may never have clear answers—absence of evidence isn't evidence of absence.

7 The Algebra of abū Kāmil

Abū Kāmil's *Kitāb fī al-jabr wa al-muqābala* was written some fifty to one hundred years after al-Khwārizmī's ground-breaking work.

At first sight it looks startlingly like al-Khwārizmī's book. The initial numerical examples are identical and so are a number of the problems. One can think of Abū Kāmil's work as an extended commentary on the algebra of his predecessor (this was once a popular genre). But there are significant differences.

- Justification of the rules for solving quadratic equations is given by explicit reference to theorems in Euclid's *Elements,* mainly from Book II (this had already been done by Thābit ibn Qurra).
- There are many more manipulational "rules of algebra."
- Several modes of attack on a problem are often given.
- Abū Kāmil sometimes uses higher powers of x.
- In a lengthy section on the pentagon and the decagon, abū Kāmil uses solutions of quadratic equations to calculate many lengths and areas. There is no mention of the inheritance problems that take up so much of al-Khwārizmī's book.
- The numbers that al-Khwārizmī uses in his problems are all rational, and answers are almost always small integers. But Abū Kāmil uses irrational numbers fairly freely. For example, he asks for two numbers whose sum is 10 (call them x and y) such that $x/y + y/x = \sqrt{5}$.
- He has many more "algebra" problems than al-Khwārizmī. Several involve quite large numbers, or remarkably extensive algebraic computation given that everything is written out in words. Here is an example that could present a bit of a challenge even now.

> One says that ten is divided into three parts, and if the small one is multiplied by itself and added to the middle one multiplied by itself, it equals the large one multiplied by itself, and when the small is multiplied by the large, it equals the middle multiplied by itself.

Abū Kāmil's work strongly influenced Leonardo of Pisa, and through him the whole Italian School of algebra. He is the last Islamic algebraist to have a demonstrated effect on algebra as it developed in Europe.

But to stop with abū Kāmil is to ignore the brilliant contributions to algebra made from the late tenth century on by al-Karajī, al-Samaw'al, Umar al-Khayyām, and Sharaf al-Dīn al-Ṭūsī. Their work goes well beyond quadratic equations; some of it will be taken up briefly later.