Part 2: Cardano's *Ars Magna*

1 Introduction

Histories of mathematics often assert that Girolamo Cardano's *Ars Magna* (The Great Art, 1545) gives algebraic solutions for all equations of degree less than or equal to four. As we shall see, from a sixteenth or even seventeenth century standpoint this is not precisely true. But the work, in the context of what was then known, is a monumental achievement, and influenced the development of algebra for a couple of centuries.

Cardano was not a modest man—he got into trouble with the Church for casting the horoscope of Christ. On the title page he describes himself as "Outstanding Mathematician, Philosopher and Physician" and the book as "... Tenth in Order of the Whole Work on Arithmetic Which is called the *Perfect Work*." The first paragraph of the Introduction reads, in part:

> In this book, learned reader, you have the rules of algebra (in Italian, the rules of the cosa). It is so replete with new discoveries and demonstrations by the author—more than seventy of them—that its forerunners ... are washed out. It unties the knot not only where one term is equal to another or two to one but also where two are equal to two or three to one.

The book ends with "Written in five years, may it last as many thousands." This hubris is characteristic of the Renaissance. A new world of possibilities had opened up, and a new aggressive individualism to go with it. Mathematicians and scientists, hitherto in awe of the work of the Ancients, began to think they could surpass it.

2 Predecessors of Cardano

2.1 Islamic Algebraists

In the twelfth century, Umar al-Khayyām tried to solve cubic equations algebraically. He didn't succeed, but produced a detailed *geometric* analysis.

A century later, Sharaf al-Dīn al-Ṭūsī made a sophisticated study of equations of type $x^3 + b = ax$ (a and b positive), and showed there is a (positive) root if and only if

$$\frac{a^3}{27} - \frac{b^2}{4} \geq 0.$$

This is the first appearance in history of the *discriminant* of a cubic. The discriminant would become crucial in the work of Cardano and his successors. Unfortunately, the results of al-Khayyām and al-Ṭūsī only reached Europe hundreds of years after Cardano.

2.2 The Italian School

There are occasional stabs at the cubic from Leonardo of Pisa on. Around 1350, Maestro Dardi looked at some special cases that reduce to $(x-a)^3 = b$. Maestro Gerardi published incorrect formulas; the errors were pointed out by Maestro Benedetto. These three men were *maestri d'abbacò,* teachers of basic commercial arithmetic and algebra. In his *Summa* of 1494, Fra Luca Pacioli mentions that a solution of the general cubic hasn't been found. The sixteenth century discoveries don't come out of thin air: solving the cubic is part of the research program of the time.

2.3 The Discoverers

Scipione del Ferro (University of Bologna, 1515?) found an algebraic solution for equations of type $x^3 + bx = c$ where b and c are positive. He communicated it to Antonio Maria Fiore, Annibale della Nave, and maybe others. But the method was known only to an in-group. A notebook of del Ferro that describes the procedure survived until at least 1570. It has disappeared.

In 1535, while preparing for a problem-solving contest against Fiore, Tartaglia also found a way to solve del Ferro's equation, and maybe $x^3 + ax^2 = d$. After pleas from Cardano, Tartaglia gave him the rule in cryptic form. Over the next few years Cardano, with the help of Lodovico Ferrrari, worked out the details for "most" cubics, and Ferrari found a method for the quartic, which however depends on the not quite digested cubic. It turns out that to understand the *real* solutions of some cubics requires knowledge of *complex* numbers. The first satisfactory treatment is by Euler around 1740; someone more fussy might say Lagrange, in 1770.

3 The Look and Feel of *Ars Magna*

3.1 Structure of *Ars Magna*

Ars Magna is structured somewhat like the theoretical part of al-Khwārizmī's *al-jabr wa al-muqābala*. Cardano acknowledges the debt: Chapter I of *Ars Magna* begins with "This art originated with Mahomet the son of Moses the Arab." More precisely, the book borrows its form from the eminent algebraist Leonardo of Pisa (1202) and from Pacioli's *Summa*.

The first ten chapters deal with linear and quadratic equations, basic algebraic techniques, and transformation methods. There follow thirteen short chapters on cubic equations, one for each of the thirteen non-trivial cases. The last seventeen chapters deal with: miscellaneous equation-solving methods; problems (many!) that lead after a while to solvable equations; proportions and the ever-popular *Rule of Three*; and some tentative explorations including seventeen pages on biquadratic (degree four) equations.

There are thirteen types of cubic for the same reason that al-Khwārizmī has three non-trivial cases of the quadratic. Only positive numbers are allowed as coefficients, so the equations $x^3 + cx = d$, $x^3 = cx + d$, $\ldots$, $x^3 + ax^2 + d = cx$ all get separate and often quite different treatment. Negative numbers are allowed as roots, albeit with some reluctance: they are called "fictitious" or "false." But Cardano is the first European mathematician to handle negative numbers with some degree of comfort. Like his predecessors, Cardano doesn't use parameters, so a, b, c and d are always particular numbers.[1]

Cardano gives lengthy geometric justification of his rules. Thus algebra is not yet an autonomous discipline: validation requires geometric proof. Euclid is often quoted.

3.2 Cardano's Notation

Leonardo of Pisa, like al-Khwārizmī, wrote everything out in words. The next three hundred years brought many abbreviations. Cardano's are a variant of the ones used by Pacioli.

The three editions of *Ars Magna* use mildly different notations. Indeed there are inconsistencies within the same page of the same edition. Most "symbols" are abbreviations of terms such as *piú* (plus), *meno* (minus) *quadratum* (square), *cubum* (cube), and so on, and abbreviations may

[1] Cardano uses parameters once. He writes that $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$ for any numbers a and b.

be short or shorter. Here are a couple of samples:

1q̄dq̄d. p: 6 q̄d. p:36 æqualia 60 pos. means $x^4 + 6x^2 + 36 = 60x$.

The q̄d stands for *quadratum*, square, so q̄dq̄d is "square-square," namely fourth power. The pos. stands for *positio*, the "thing" which is sought. Most Italian mathematicians called it *res* or *cosa*.[2] In some formulas, *æqualia* (equals) becomes æqual., æq̄lia, æq̄, or even just a long blank space.

The letter R (more precisely, the pharmacy prescription sign) stands for *radix* (root). The expression

R v: cu. R 108 m: 10 means $\sqrt[3]{\sqrt{108} - 10}$.

The R 108 m: 10 is just $\sqrt{108} - 10$. R cu. means *radix cubica*, cube root. The v in between is the first letter of *universalis.* It means that you take the cube root of all that follows. The parentheses we now use for grouping are very rare until the eighteenth century.

We note here that Italy was quite conservative in notation, maybe because it had a long mathematical tradition. Forms of our "+," "−," "$\surd$," and "=" were already being used by some German, Dutch, and English writers.

3.3 Cardano Translated

Cardano's book, like a majority of scholarly works up to the nineteenth century, was written in Latin. An English translation was published in 1968. This short excerpt from Chapter XXXVII is unrepresentative, since it deals with a quadratic equation, but historically important. It is the first time that a complex number is used in mathematics.

> If someone says to you, divide 10 into two parts, one of which multiplied into the other shall produce 30 or 40, it is evident that this case or question is impossible. Nevertheless, we shall solve it in this fashion. Let us divide 10 into equal parts and 5 will be its half. Multiplied by itself, this yields 25. From 25 subtract the product itself, that is 40, which as I taught you . . . leaves a remainder m: 15. The root of this added and then subtracted from 5 gives the parts which multiplied together will produce 40. These, therefore, are 5 p: R m: 15 and 5 m: R m: 15.

[2] In English, algebra was called the art of the coss, and practitioners were called cossists.

Cardano makes a doomed stab at a geometric justification, and ends up formally computing $(5+\sqrt{-15})(5+\sqrt{-15})$. He calls the quantity he has introduced *sophistica* (sophistic, unreal), and the arithmetic he has done "as subtle as it is useless." It took another 200 years for complex numbers to be fully integrated into mathematics.

4 Solving Cubic Equations

Our presentation in modern notation will be far shorter than the 59 pages Cardano devotes to the thirteen cases, and the many pages of preliminaries. There are a number of reasons for this.

Cardano's presentation is very detailed, with many numerical examples. He is, by our standards, quite wordy: current fashion favours a more telegraphic style. And of course modern notation is very concise. The conceptual base has changed. Since we take the "rules of algebra" for granted, there are no geometric justifications. Since we allow 0 and negative numbers as coefficients, there is only one case. We don't care whether a root is positive, and Cardano always does. But our mechanical manipulations don't do justice to the subtle intelligence and complexity of Cardano's work.

4.1 Preliminary Reduction

We use modern notation. But we need to be aware that for Cardano "x" represents a particular but unknown number: he does not have the concept of *polynomial.*

Let $P(x) = x^3 + ax^2 + bx + c$ where a, b, and c are real. We want to solve the equation $P(x) = 0$. (The first person to allow letters a, b, ... to represent a number that could be positive or negative seems to be Hudde, around 1659.)

Let $x = y - a/3$. We are studying the equation

$$(y - a/3)^3 + a(y - a/3)^2 + b(y - a/3) + c = 0. \tag{1}$$

Simplify. It is easy to see that the coefficient of y^2 in (1) is 0, so the equation has shape

$$y^3 + py + q = 0 \tag{2}$$

where p and q aren't hard to find. The roots of $P(x) = 0$ are the roots of (2) shifted by $a/3$.

The substitution $x = y - a/3$ seems to be due to Cardano. Actually, he uses $x = y - a/3$ if the x^3 and x^2 terms are on the same side of the equation, and $x = y + a/3$ if they are not. How could del Ferro and Tartaglia have missed it? Centuries before, al-Khayyām and al-Ṭūsī used related transformations in their study of cubics.

4.2 The Substitution $y = u + v$

The following identity is easy to verify:

$$(u+v)^3 - 3uv(u+v) - (u^3+v^3) = 0 \tag{3}$$

In essence Cardano uses the same identity. He justifies it by a volume calculation, dissecting actual cubes. Cardano was undoubtedly guided by the standard "completing the square" arguments for the quadratic. Cardano needs u and v to be sides of a cube. In some cases that forces him to use the substitution $y = u - v$—that's how he solves his first serious cubic equation, $x^3 + 6x = 20$.

Identity (3) shows that if

$$3uv = -p \qquad \text{and} \qquad u^3 + v^3 = -q \tag{4}$$

then $u + v$ is a solution of $y^3 + py + q = 0$.

Can we find u and v that satisfy the equations of (4)? *Assume* there is a solution, and on that assumption, use algebraic manipulations to find one. Finally—important—either verify that the solution works, or that the steps in the argument are reversible. For though all humans are bipeds, not all bipeds are human—some bipeds are pheasants.

Write $v = -p/3u$ and substitute. After some manipulation we arrive at

$$(u^3)^2 + qu^3 - \left(\frac{p}{3}\right)^3 = 0, \tag{5}$$

So u^3 is a solution of the quadratic equation $z^2 + qz - (p/3)^3 = 0$. The solutions are A and B where

$$A = -\frac{q}{2} + \sqrt{(q/2)^2 + (p/3)^3} \qquad \text{and} \qquad B = -\frac{q}{2} - \sqrt{(q/2)^2 + (p/3)^3}.$$

It is easy to check by using the "rules of algebra" that $AB = -(p/3)^3$ and $A + B = -q$. But we may need to enter the world of *complex* numbers—the stuff under the square root sign could be negative. Let $\Delta = 4p^3 + 27q^2$. The number Δ would later be called the *discriminant* of the equation.

Cardano is quite aware of the role of the discriminant. In Chapter XII (*On the Cube Equal to the First Power and Number*, $x^3 = px + q$) he quickly shows how to deal with the problem when "the cube of one-third the coefficient of x is not greater than the square of one-half the constant of the equation," that is, when $\Delta \geq 0$. For the case $\Delta < 0$, he writes "…you may, for the most part, be satisfied by Chapter XXV" (*Imperfect and Particular Rules*). But Chapter XXV is not at all satisfactory. There he finds solutions only in some *very* special cases, and makes vague proposals that hint at methods but in fact don't work.

4.3 Cardano's Formula

Assume that $\Delta \geq 0$. Let $u = \sqrt[3]{A}$ and $v = \sqrt[3]{B}$. Then (u, v) is a solution of the system (4). So one solution of $y^3 + py + q = 0$ is

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{\Delta}{108}}} \quad + \quad \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{\Delta}{108}}}. \tag{6}$$

Formula (6) is now called *Cardano's Formula.* All of the above, in very different form[3] of course, can be found in *Ars Magna.* Cardano can't handle negative discriminants and he knows it: his thirteen rules work, in his sense, only for equations that reduce to an equation with $\Delta \geq 0$.

Whenever $\Delta > 0$, the remaining two roots of the cubic are not real, so Cardano's Formula gives the "only" (real) solution. When $\Delta < 0$, Cardano's Formula is (for us) a correct expression for one of the roots. To Cardano, his method is simply inapplicable in this case, which came to be known as the *casus irreducibilis* and remained for many years a source of puzzlement.

In dealing with some of the types, Cardano doesn't use the method described above: instead, he reduces the problem to a type solved earlier. For example, he gives (and proves) the following rule for the type $x^3 + q = px$.

Find a solution y of the equation $y^3 = py + q$. Then

$$x = y/2 + \sqrt{p - (y/2)^2}$$

is a solution of $x^3 + q = px$.

4.4 The Discriminant and the Nature of the Roots

When $\Delta = 0$ we can easily find all of the roots. Let $\sqrt[3]{q/2} = r$. Then $q = 2r^3$ and $p = -3r^2$. But

$$y^3 - 3r^2 y + 2r^3 = (y - r)(y - r)(y + 2r),$$

[3] Cardano would have some trouble recognizing the formula named after him. The same is true for de Moivre, Taylor, and many others.

so the roots are real, with r a "double" root. (The concept of multiple root first comes up in the work of Girard (1629), in connection with the Fundamental Theorem of Algebra.)

Cardano knew that if $\Delta > 0$ there is only one (real) root. He was also more or less aware that when $\Delta < 0$ there are three (real) roots, though he does not prove it. These facts *can* be proved using techniques already present in the work of al-Ṭūsī, some 300 years before Cardano. We give a quick exposition using first-year calculus. That's not as anachronistic as it looks: al-Ṭūsī had an embryonic algebraic version of the derivative!

Let $f(x) = x^3 + px + q$. Suppose first that $p > 0$. Then $f(x)$ is an increasing function, so it is 0 exactly once. Note that $\Delta > 0$.

Suppose next that $p = 0$. Then $f(x)$ is 0 only once. We have $\Delta > 0$ unless $q = 0$. In that case 0 is a triple root.

Finally, let $p < 0$. Then Δ *could* be negative. Temporarily, let $p = -3k^2$, where $k > 0$. Then $f(x) = x^3 - 3k^2x + q$ and $f'(x) = 3x^2 - 3k^2$. Thus $f'(x) = 0$ when $x = \pm k$, and $f(x)$ is increasing on the interval $(-\infty, -k)$, decreasing on $(-k, k)$, and increasing on (k, ∞). There is a local maximum at $x = -k$ and a local minimum at $x = k$. A little playing with pictures shows that there are three distinct real roots iff $f(-k) > 0$ but $f(k) < 0$. These two inequalities are equivalent to $f(-k)f(k) < 0$. But

$$f(-k) = 2k^3 + q \qquad \text{and} \qquad f(k) = -2k^3 + q$$

and therefore

$$f(-k)f(k) < 0 \qquad \text{iff} \qquad -4k^6 + q^2 < 0.$$

Since $k^6 = -p^3/27$, equivalently we have $4p^3 + 27q^2 < 0$, that is, $\Delta < 0$.

We have shown that the discriminant is negative, meaning that in Cardano's Formula square roots of negative numbers are unavoidable, *precisely* when all the roots of the cubic are distinct and real! It is this fact that first forced later mathematicians to take square roots of negative numbers seriously. With quadratic equations, the situation is different: there, square roots of negative numbers are simple to avoid—just conclude that the equation has no solution.

5 Ferrari's Method for the Quartic

5.1 The Original Method

The material in this section can be found in *Ars Magna.* The details are less fully fleshed out than for the cubic. Cardano credits the idea to Ferrari, a gifted young man who came to work with him. Though uncomfortable with the considerable distortion this entails, we describe Ferrari's method in modern notation.

Given the equation $x^4 + ax^3 + bx^2 + cx + d = 0$, put $x = z - a/4$ and simplify. We obtain a quartic equation in z with the z^3 term missing. Since it's nice to have x as the name of the unknown, assume that we started with $x^4 + px^2 + qx + r = 0$.

Ferrari's idea is to produce the "complete" square

$$\left(x^2 + \frac{p}{2} + y\right)^2$$

on the left-hand side. So he rewrites the equation as

$$\left(x^2 + \frac{p}{2} + y\right)^2 = 2yx^2 - qx + \left(\frac{p}{2} + y\right)^2 - r. \tag{7}$$

(It would have been a bit simpler to have $(x^2 + y)^2$ on the left, with appropriate changes on the right.)

Ferrari chooses y so that the polynomial on the right-hand side is a perfect square $(sx + t)^2$ where s and t are numbers. This is equivalent to asking that the polynomial not have distinct roots. And that is the case iff the discriminant (in the usual school sense) is 0, that is,

$$q^2 - (4)(2y)\left[\left(\frac{p}{2} + y\right)^2 - r\right] = 0. \tag{8}$$

Equation (8) is called the *resolvent equation*; it is cubic in y. Find a real root y. Now that y is determined, we can find s and t such that

$$\left(x^2 + \frac{p}{2} + y\right)^2 = (sx + t)^2. \tag{9}$$

The rest is easy. Equation (9) holds iff

$$x^2 + \frac{p}{2} + y = sx + t \qquad \text{or} \qquad x^2 + \frac{p}{2} + y = -(sx + t).$$

The two quadratic equations are straightforward to solve.

5.2 A Variant of Ferrari's Method

It was natural for Ferrari to recycle Cardano's idea and first knock out the x^3 term. But we can work directly with $x^4 + ax^3 + bx^2 + cx + d = 0$. Rewrite this as

$$\left(x^2 + \frac{ax}{2} + y\right)^2 = 2yx^2 + \frac{a^2}{4}x^2 + ayx + y^2 - (bx^2 + cx + d).$$

The right-hand side is a perfect square iff the discriminant is 0, so as with Ferrari's method we need to solve a cubic equation in y. This version is given in a 1740 book by Simpson. There is nothing of substance new in it.