Part 3: Cubics, Trigonometric Methods, and Angle Trisection

1 Trigonometric Solution of the *Casus Irreducibilis*

1.1 Introduction

The following argument is in essence due to François Viète (1540–1603). The result was published in 1615.[1] Albert Girard (1595–1632) was the first explicitly to use a version of identity (1) to solve cubic equations. In his *L'Algebra* (1572) Bombelli gives an ingenious geometric solution of the irreducible case. He is aware that his method is connected to ancient Greek methods for trisecting the angle and speculates that there is a deeper connection. But he doesn't work out the details, so misses the trigonometric solution. For simplicity, this chapter uses modern notions and notations.

1.2 A Trigonometric Identity

By the usual formula for the cosine of a sum,

$$\cos 3\alpha = \cos(2\alpha + \alpha) = \cos 2\alpha\, \cos\alpha - \sin 2\alpha\, \sin\alpha.$$

But

$$\begin{aligned} \cos 2\alpha\, \cos\alpha &= (2\cos^2\alpha - 1)\cos\alpha = 2\cos^3\alpha - \cos\alpha \qquad \text{and} \\ \sin 2\alpha\, \sin\alpha &= (2\sin\alpha\, \cos\alpha)\sin\alpha = (2\cos\alpha)(1 - \cos^2\alpha). \end{aligned}$$

Putting everything together, we get the important identity

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha. \tag{1}$$

This identity is central to the solution of the famous problem of trisecting the general angle with straightedge and compass. We deal with that later in the chapter.

[1] The good ones keep on producing even after they are dead.

Comment. It is not hard to show that for any integer $n \geq 0$ there is a polynomial $T_n(t)$ of degree n such that $\cos n\alpha = T_n(\cos \alpha)$. The first five of these polynomials are 1, t, $2t^2 - 1$, $4t^3 - 3t$ (we have just proved this), and $8t^4 - 8t^2 + 1$.

Polynomials related to these were first studied by Viète, so naturally they are called the *Chebyshev polynomials*, after the nineteenth-century Russian mathematician.[2]

1.3 Solving the Irreducible Case

We show how identity (1) can be used to "solve" a particular cubic equation, and then generalize to all *casus irreducibilis* cubics.

1.3.1 A Special Equation

We solve the equation $4x^3 - 3x = -\frac{1}{2}$. Look for a solution of the form $x = \cos \alpha$. Then $4x^3 - 3x = \cos 3\alpha$, so by identity (1) we are looking at the equation $\cos 3\alpha = -1/2$.

Recall that $\cos 120° = -1/2$. So we can let $3\alpha = 120°$, that is, $\alpha = 40°$, and therefore the equation has $x = \cos 40°$ as a solution.

What about other solutions? There are a couple of approaches. For brevity, let $c = \cos 40°$. Then $4c^3 - 3c = -1/2$. So our original equation is equivalent to $(4x^3 - 3x) - (4c^3 - 3c) = 0$, that is, $4(x^3 - c^3) - 3(x - c) = 0$. By factoring, we can rewrite this as

$$(x - c)[4(x^2 + cx + c^2) - 3] = 0,$$

and now we need only solve a quadratic equation.

There is a better way. Angles other than $120°$ have cosine equal to $-1/2$, for example $240°$ and $480°$. We conclude that $x = \cos 80°$ and $x = \cos 160°$ are solutions.

Using a calculator, we can show that $\cos 40°$, $\cos 80°$, and $\cos 160°$ are all different. Without a calculator it's even easier. A cubic has no more than three roots, we found three, so we got them all.

The root $x = \cos 40°$ is really $x = \cos((\arccos(-1/2))/3)$. This is *not* an "algebraic" expression, for it uses the transcendental functions arccos and cos. For our purposes, an algebraic expression must only involve the

[2] Why T_n? His name used to be transcribed from the Cyrillic to the Latin alphabet as, among other things, *Tchebycheff*. The Chebyshev polynomials have many engineering applications, for example in loudspeaker design, the design of electrical filters, and the analysis of heat flow.

usual operations of arithmetic and n-th roots. The usual term for the kind of solution we are looking for is *solution by radicals* (here radicals means square roots and/or cube roots and/or ...).

Comment. We use degree notation for two reasons. "Everyone" did up to the time of Euler. For computations, degrees were standard up to the calculator age; they are still standard in the real world. And using degrees connects us with the Babylonian roots of the subject. The division of degrees—and hours—into minutes and seconds[3] comes from the Babylonian (modified) base 60 system for writing numbers, so it is in principle about 4000 years old.

1.3.2 The General Case

Look at the cubic $y^3+py+q=0$, and suppose that $p<0$ (the discriminant Δ can only be negative if p is). Let $p=-3k^2$. By an appropriate change of variable, we want to make the cubic look like $4z^3-3z=e$.

Multiply $y^3-3k^2y+q=0$ through by 4, and let $y=rz$, where r will be chosen in a minute. Substitute for y and divide through by r^3. We get

$$4z^3-\frac{12k^2}{r^2}z=-\frac{4q}{r^3}.$$

Now let $r=2k$. We end up with

$$4z^3-3z=-\frac{q}{2k^3} \tag{2}$$

Let the right-hand side be e. If $|e|\leq 1$, then $e=\cos 3\alpha$ for some angle 3α. So by identity (1), $z=\cos\alpha$ is a solution of equation (2).

To find all solutions, note that $\cos 3\alpha=\cos(360^\circ-3\alpha)=\cos(360^\circ+3\alpha)$. It follows that

$$\cos\alpha, \qquad \cos(120^\circ-\alpha) \qquad \text{and} \qquad \cos(120^\circ+\alpha) \tag{3}$$

are solutions.

The above procedure works whenever e is the cosine of something, that is, when $e^2\leq 1$. But $e^2=q^2/4k^6=-27q^2/4p^3$, so $e^2\leq 1$ if and only if $4p^3+27q^2\leq 0$. It follows that the procedure works in the irreducible case. Except in the trivial cases $e=\pm 1$, the solutions given in (3) are all different, so we have found trigonometric expressions for all the solutions.

[3] The word minute comes from *pars minuta* (small part), and second comes from *pars minuta secunda*. For high precision work, astronomers used *pars minuta tertia,* and so on.

1.3.3 Viète's Trigonometric Solution

Now that we have done it in "modern" (post-Euler) style, we describe more exactly Viète's version of the trigonometric solution. Let $\mathcal{C}$ be a fixed circle of radius 1 (Viète didn't use 1). Let P be a point on the circle. Starting at P, travel counterclockwise around the circle through an arc of length δ. Let Q be the point we end up at. Then the distance PQ is called the *chord* of δ; we abbreviate this as $\operatorname{crd}\delta$.

The "chord" function goes back to Hipparchus (about -150), and perhaps earlier. We will study it in more detail when we deal with the history of trigonometric functions: it is a relative of the sine function. Instead of using identity (1), Viète used the fact that

$$\operatorname{crd} 3\delta = 3 \operatorname{crd}\delta - \operatorname{crd}^3 \delta. \tag{4}$$

Identity (4) is (for us) not hard to verify by using the fact that $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$. Viète used a geometric argument that wasn't really new—a closely related result had been published not long before by Pitiscus (1561–1613). And Ptolemy (c. 100–c. 170) made frequent use of a result equivalent to our usual formula for $\sin(\alpha+\beta)$, so the tools needed to produce identity (4) had been available for a long time.

Around 1400, al-Kāshī was computing trigonometric tables and needed to find $\sin 1^\circ$ to high accuracy. Al-Kāshī knew a formula equivalent to $\sin 3\alpha = 3\sin\alpha - 4(\sin\alpha)^3$. (Some earlier Islamic mathematicians may also have known it.) Using impressively efficient numerical techniques, al-Kāshī obtained a startlingly good estimate of $\sin 1^\circ$ by looking at a variant of the equation $3x - 4x^3 = \sin 3^\circ$. (There is a trick that goes back to Ptolemy for calculating $\sin 3^\circ$.)

Comment. By the time of Viète, high accuracy trigonometric tables were available—Viète himself contributed to their improvement. So the trigonometric method was viewed as a good *computational* procedure for finding the roots of a *casus irreducibilis* cubic. Now we would probably use an equation-solving package based on some variant of Newton's Method.[4]

[4] I was once asked a question that came down after a while to solving cubics of the *casus irreducibilis* type. Naturally, I suggested Newton's Method. That wasn't suitable, since the person needed a *formula,* for use with an early spreadsheet, so I used the trigonometric method. There was even some money in it—not much, but enough for a fair number of bottles of cheap wine. And there was pleasure in seeing an antique formula, probably not used for practical purposes for many years, turn out to be useful.

1.3.4 Positive Discriminants

We saw that the trigonometric procedures work in the *casus irreducibilis.* What about the "ordinary" case? We can use the Cardano Formula, but it is interesting to develop an analogue of the trigonometric method. We can do that by using appropriate identities for $\cosh 3\alpha$ and $\sinh 3\alpha$.

2 Cardano's Formula and $4x^3 - 3x = -\frac{1}{2}$

Cardano's Formula, applied mechanically, says that

$$\sqrt[3]{\frac{-1+\sqrt{-3}}{16}} + \sqrt[3]{\frac{-1-\sqrt{-3}}{16}} \tag{5}$$

is a root of the equation $4x^3 - 3x = -1/2$. Once we understand complex numbers well enough, it will turn out that (5) is a correct expression for one of the roots. The Cardano Formula, properly understood, is correct even in the irreducible case. So if we are allowed to use cube roots of complex numbers we have a formula for one of the roots, and with some modification all the roots, of any cubic.

That raises the following question: is there an expression for one of the roots of $4x^3 - 3x = -1/2$ that uses rational numbers, addition, subtraction, multiplication, division, and perhaps square roots, cube roots, fourth roots, fifth roots, ... of *real* numbers? After all, the roots are all real, so why should we have to travel through the complex numbers to get a formula for them? If we use the Cardano Formula, then we do travel through the complex numbers, but might there be another formula? Can we get a formula using *real radicals*? The question is answered by the following result.

Theorem 1. *Consider the equation $x^3 + px + q = 0$, where p and q are rational numbers and the discriminant Δ is negative. If this equation has no rational solutions, then any expression in terms of radicals for any solution* must *involve a root of a (non-real) complex number.*

The proof is unfortunately much too hard to present here. It involves sophisticated tools from *Galois Theory.* I have only seen twentieth century proofs, but suspect that the first proof was given in the late nineteenth century.

It is not very hard to show that the equation $4x^3 - 3x = -1/2$ has no rational solutions, so by the above theorem any "algebraic" expression for a root, and therefore any algebraic expression for $\cos 40^\circ$, must travel through the complex numbers—they are unavoidable, even though the end result is real.

3 Trisecting Angles and Related Matters

3.1 Introduction: Three Famous Problems of Antiquity

The following geometric problems are very old:

The Duplication of the Cube: Given a cube, construct a cube with twice the volume.[5]

The Trisection of Angles: Given an angle, divide that angle into three equal parts.

Squaring the Circle: Given a circle, construct a square with the same area.[6]

The mathematicians who made contributions to the three problems include a virtual who's who of Greek geometry.[7] More recently, contributions

[5]The story is told that during a plague, an oracle informed the Delians that their sufferings would end if they constructed a new altar to Apollo *exactly* twice as big as the current one. They doubled all the dimensions, but people kept dying. So they realized that it was the *volume* that needed doubling. One version of the story stops here, with no explanation of why the gods need to have an altar doubled, and if so why they don't do it themselves. Another version has the Delians asking for Plato's help. Plato replies that the problem can undoubtedly be solved by one of the geometers in the Academy, and tells the Delians that the gods don't *really* want the altar doubled, they just want to punish the Greeks for neglecting geometry.

The story is presumably false, for Hippocrates of Chios made an important technical contribution to cube duplication when Plato was at most a small child. But it correctly portrays Plato's attitude to mathematics. "Let no one ignorant of geometry enter these doors" is said to have been inscribed above the entrance to the Academy. Is this the first Mathematics Entrance Requirement?

[6]Aristophanes makes a comic reference to circle squaring in *The Birds*, first staged in -414. It is also said that the philosopher Anaxagoras worked on the problem while in prison around -450. Anaxagoras is not the only person to do mathematics in jail. Poncelet made major contributions to projective geometry while a prisoner of the Russians during the Napoleonic wars. More recently, we have André Weil, mathematically productive in a French jail early in the Second World War.

[7]Here is a partial list: Apollonius, Archimedes, Archytas, Dinostratus, Diocles, Eratosthenes, Eudoxus, Hero, Hippocrates, Menaechmus, Nicomedes, Pappus, Philo. (Euclid seems to have been immune.)

have come from Viète, Descartes, Pascal, Huygens, Newton, Gauss, Wantzel, Kempe, Lindemann, and many others.

"Construct" somehow has come to mean "construct with straightedge and compass," but the Greeks had no such restriction, and devised an impressive menagerie of curves and mechanical devices for the express purpose of effecting the constructions. The only thing they insisted on is that the constructions be exact.[8]

Pappus (c. 350) argued, using completely muddled reasoning, that the three constructions cannot be done by straightedge and compass. But the *proof* that this is the case requires algebraic ideas that the Greeks did not possess. In particular, the link between geometric figures and algebraic equations, developed in the seventeenth century by Fermat and Descartes, is an essential element of the proof. In 1837, Wantzel published a proof that the cube cannot be duplicated and the general angle cannot be trisected with straightedge and compass.

In 1882, Lindemann proved that π is *transcendental*, meaning that π is not a root of a non-trivial polynomial with integer coefficients. This implies that the circle cannot be squared with straightedge and compass, and in a sense brings closure to a 2300 year old problem.[9]

3.2 The Trisection of Angles

We sketch the connection between the trisection problem and cubic equations. Note first that *some* angles can be trisected with straightedge and compass. For example, a 90° angle can be: just use straightedge and compass to construct a 30° angle.

We can certainly construct a 120° angle. Thus if we had a straightedge and compass trisection of the general angle, we could construct a 120° angle,

[8] It is not hard to find procedures for solving the problems *approximately* to any desired degree of precision. So the task of solving the problems *exactly* is of no direct applied interest, unless the application involves pleasing the gods, who can be awfully fussy.

[9] The discovery that the three famous constructions can't be done with straightedge and compass did not stop the enthusiastic armies of circle squarers and angle trisecters. Because of the presence of the famous geometer H.S.M. Coxeter, constructions (usually trisections) often were sent to the University of Toronto. The paper would be handed to a luckless student, who was asked to find the first error. But rejection doesn't stop determined trisecters or circle squarers, and many of their 'discoveries' have been privately published. For an entertaining account of this underside of mathematics, see *Mathematical Cranks* by Underwood Dudley.

There are fewer angle trisecters around than there used to be—certainly it is a while since a trisection has been submitted to the Mathematics Department. This is not an entirely healthy sign: it may be symptomatic of our lamentable neglect of geometry.

then trisect it to produce a 40° angle. But in fact a 40° angle *cannot* be constructed using straightedge and compass.

3.2.1 Preliminary Reduction

Let's first define (roughly) what it means for a *number* to be constructible. Initially, we are given the points $O = (0, 0)$ and $X = (1, 0)$ in the coordinate plane. We say that the number α is *constructible* if starting from O and X, we can using straightedge and compass produce the point with coordinates $(\alpha, 0)$. It is easy to see that if α and β are constructible, so are $\alpha \pm \beta$. By playing wiith similar triangles, one can show that $\alpha\beta$ is also constructible, as are α/β and $\sqrt{\alpha}$ (if they exist).

Next we observe that an angle θ (say less than a right angle) can be constructed if and only if the number $\cos\theta$ is constructible. For suppose that we could make an angle θ, i.e., construct three points A, B, and C such that $\angle BAC = \theta$. Drop a perpendicular from B to the line AC, meeting that line at say D. We have $\cos\theta = AD/AB$. Certainly the lengths AD and AB can be constructed, and thus so can their ratio $\cos\theta$. It is equally straightforward to show that if $\cos\theta$ is constructible, then we can construct points B, A, and C such that $\angle BAC = \theta$.

In view of the above observations, in order to show that a 40° angle cannot be constructed with straightedge and compass (and hence that the 120° is not trisectable with these tools), it is enough to show that the number $\cos 40^\circ$ is not constructible.

Comment. The fact that an angle is constructible iff its cosine is a constructible number is very simple to establish. But one should be aware that it is a remark that could not be made before the seventeenth century, when workers, led by Fermat and Descartes, began to identify the ancient Euclidean plane with the collection of all pairs (a, b) where a and b are real numbers.

3.2.2 Showing that $\cos 40^\circ$ is not Constructible

We saw in Section 1.3.1 that the number $\cos 40^\circ$ is a solution of the cubic equation $4x^3 - 3x + 1/2 = 0$. We will use the following result:

Theorem 2 (Wantzel, 1837). *Let $P(x)$ be a cubic polynomial with rational coefficients, and suppose that the equation $P(x) = 0$ has no rational roots. Then no root of $P(x) = 0$ is constructible by straightedge and compass.*

The proof of this result is not particularly deep, but it is fairly long, and we omit it. One first shows that the constructible numbers are precisely the numbers that can be produced by the operations of elementary arithmetic together with square root. Then one shows that roots of the cubics of Theorem 2 can't be so produced.

Note that Theorem 2 settles the trisection problem. For it is not hard to show that $4x^3 - 3x = -1/2$ has no rational solutions, and therefore $\cos 40°$ is not constructible, so the 120° angle is not trisectable by straightedge and compass.

Theorem 2 also settles the problem of duplicating the cube. For given a cube of side 1, in order to duplicate it we must construct the number $\sqrt[3]{2}$, which is a root of the equation $x^3 - 2 = 0$. But this equation has no rational roots, and therefore $\sqrt[3]{2}$ is not constructible.

3.2.3 Constructible Regular Polygons: The Gauss-Wantzel Theorem

A subject even more ancient than the ones we have been discussing is the straightedge and compass construction of the regular polygons. (A convex polygon is *regular* if all sides and all angles have the same size. An n-gon is a polygon with n sides.)

Clearly we can construct a regular 3-gon (equilateral triangle) and a regular 4-gon. For any regular n-gon, the angle subtended at the centre by one of the sides is $360°/n$. It is easy to see that if a regular n-gon is constructible, so is the angle $360°/n$, and conversely.

Let $n = 5$. It is not hard to show that the regular pentagon is constructible.[10] The case $n = 6$ is easy, as is $n = 8$.

The case $n = 7$ is more complicated. It turns out that the regular heptagon is *not* constructible by straightedge and compass. (The proof uses Theorem 2.) But a "construction" using other tools is traditionally ascribed to Archimedes, and was definitely done by Thābit ibn Qurra (909–946).

Let $n = 9$. Any side of the regular 9-gon subtends a 40° angle at the centre of the 9-gon. But the 40° angle is not constructible. and therefore the regular 9-gon is not (straightedge and compass) constructible.

The following result more or less settles the question of which regular

[10]Euclid gives a straightedge and compass construction for the regular pentagon in Book IV of the *Elements*, but the early Pythagoreans probably already knew how to do this. The regular pentagon and pentagram had symbolic significance for them.

n-gons are constructible. First define the *Fermat numbers* F_k by the formula

$$F_k = 2^{2^k} + 1 \qquad \text{for } k = 0,\, 1,\, 2,\, \ldots.$$

The first five Fermat numbers, corresponding to $k = 0$, 1, 2 , 3, and 4, are 3, 5, 17, 257, and 65537.

These five Fermat numbers are all prime. Fermat (1601–1665) conjectured that F_n is prime for all n—well, he did more than conjecture, at one point he wrote that he had a proof. Euler about 100 years later showed that F_5 is not prime. Indeed no F_n beyond F_4 is known to be prime. If F_n is prime, it is called a *Fermat prime.* The problem of which regular n-gons are constructible is settled by the following result, whose proof is unfortunately much too complicated to give here.

Theorem 3 (Gauss, Wantzel). *Let $n \geq 3$. The regular n-gon is constructible if and only if n is of the form $2^a p_1 p_2 \cdots p_k$, where the p_i are* distinct *Fermat primes.*

In 1796, Gauss showed that the regular 17-gon is constructible by straightedge and compass. This was the first bit of progress on constructibility of regular polygons since the ancient Greeks. Gauss shortly after that proved that p-gons where p is a Fermat prime are constructible, and from this it follows easily that all regular n-gons where $n = 2^a p_1 p_2 \cdots p_k$, where the p_i are distinct Fermat primes, are constructible. In *Disquisitiones Arithmeticae* (1801), Gauss claimed that he could prove the converse. The first published proof was by Wantzel in 1837.[11] Wantzel's proof was not quite up to later standards of rigor: some say that the first proof was by Pierpont[12] in 1895.

Analysis of regular polygons becomes more natural once we have some background in complex numbers; we return briefly to these matters in a later chapter.

[11] It is said that when Gauss (who was then 18!) found a construction for the regular 17-gon, he decided to do mathematics rather than philology, and that towards the end of his life he asked that the regular 17-gon be inscribed on his tomb. It wasn't; anyway, a regular 17-gon carved in stone would look pretty much like a circle. (Archimedes is said to have asked that a sphere inscribed in a cylinder be placed as a monument on *his* tomb, and although he was killed in the Roman sack of Syracuse, his request seems to have been carried out.)

[12] Pierpont certainly thought so. He was an American, and probably wasn't even aware of Wantzel.