Part 5: The Theory of Equations from Cardano to Galois

1 Cyclotomy

1.1 Geometric Interpretation of Complex Numbers

We are now accustomed to identifying the complex number $a + ib$ with the point (a, b) of the coordinate plane. Under this identification, $(a + ib)(\cos\theta + i\sin\theta)$ is the complex number $c + id$, where (c, d) is obtained by rotating (a, b) counterclockwise about the origin through an angle θ. This geometric interpretation is very useful, indeed indispensable, and does much to demystify complex numbers.

It took an astonishingly long time to get there. There were too many years of blind manipulation. For instance, Leibniz early in his career was naively proud about having shown that $\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}$. But the computation is trivial and the result false for all but one of the 4 possible interpretations of the objects on the left side.[1]

Euler and others sometimes think of a complex number as a point in the plane. What's missing is the interpretation of multiplication. Wessel published a clear exposition of the full geometric interpretation in 1799, but it was written in Danish, he wasn't a professional mathematician, his tone was too modest, and nobody paid attention.

The idea comes up again in an anonymous 1806 pamphlet by Argand, whose idea is then borrowed by Français. Argand gets back into the game, and there are arguments over whether it is proper to mix algebraic and geometric ideas! (The right answer is that it's stupid not to.) Mainstream mathematics still pays no attention. Gauss probably thought of complex numbers in geometric terms by 1800. His first paper that explicitly uses the idea is in 1831. Why wait 31 years? One can speculate that complex numbers were still not quite respectable in 1800, and Gauss didn't like to

[1] Puzzle: by the rules of algebra $\sqrt{-4}\sqrt{-4}=-4$; but by the rules of algebra $\sqrt{-4}\sqrt{-4}=\sqrt{16}=4$. Even the great Euler, in his *Algebra*, used contradictory "rules of algebra."

stick his neck out.[2]

In 1835, Hamilton gives a purely arithmetical definition[3] of complex number that is really the same as the geometric interpretation, but has a more modern feel. He defines a complex number as an ordered pair (a, b) where a and b are real. Addition is done in the obvious way, while the product $(a, b)(c, d)$ is defined to be $(ac - bd, ad + bc)$. The complex number $(a, 0)$ is to be informally identified with the real number a. Note that $(0, 1)(0, 1) = (-1, 0)$: finally, -1 has a square root. With the geometric interpretation or its arithmetical version, complex numbers have become real. It only took 250 years to get there.

1.2 Roots of Unity and Regular Polygons

Even without interpreting multiplication, once we have a formula for the n-th roots of unity we can think of them as the vertices of a regular n-gon inscribed in the circle $x^2 + y^2 = 1$, with one vertex at $(1, 0)$. With the geometric interpretation of multiplication, the formula for the roots of unity becomes obvious. For it is clear that if the point $(1, 0)$ is rotated n times through an angle of $2\pi k/n$, we are back where we started. The study of the n-th roots of unity is called *cyclotomy* (circle division).

Cyclotomy dates back to the Greek interest in regular polygons. When abū Kāmil used quadratic equations to calculate lengths connected with the regular pentagon and decagon, he was doing cyclotomy. So was Thābit ibn Qurra when he studied the regular heptagon. Cyclotomy was an active research subject in the eighteenth century. But up to the time of Euler the focus of attention was on either $\cos(2\pi k/n)$ or $\sin(2\pi k/n)$, and not on the algebraically (and geometrically) much more natural $\cos(2\pi k/n) + i\sin(2\pi k/n)$, that is, the n-th roots of unity.

1.3 The Pentagon and the Heptagon

1.3.1 The Pentagon

We can handle the pentagon without mentioning roots of unity—Euclid did it geometrically, and abū Kāmil gave an algebraic treatment. But the approach through complex numbers is instructive.

The polynomial $x^5 - 1$ factors as $(x - 1)(x^4 + x^3 + x^2 + x + 1)$. We could solve $x^4 + x^3 + x^2 + x + 1 = 0$ by using say Ferrari's procedure. Instead we use

[2]If we are to believe Gauss, he explored Non-Euclidean Geometry well before Bolyai and Lobachevsky, but didn't publish.

[3]Gauss had found it earlier; Hamilton was first to publish.

a trick of de Moivre. Our equation is a *reciprocal equation*: its coefficients read the same from left to right as they do from right to left.[4]

Divide both sides of the equation by x^2. After rearranging we get

$$x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0.$$

Note that $x^2 + 1/x^2 = (x + 1/x)^2 - 2$, and let $y = x + 1/x$. Our equation becomes $y^2 + y - 1 = 0$, which has the roots $(-1 \pm \sqrt{5})/2$. Now solve the disguised quadratics $x + 1/x = (-1 \pm \sqrt{5})/2$.

There is no need to write down the solutions. For the roots of $x^5 - 1 = 0$ are given by $x = \cos(2\pi k/n) + i \sin(2\pi k/n)$ as k ranges from 0 to 4. Let $\phi = 2\pi k/5$. Then

$$\frac{1}{x} = \frac{1}{\cos\phi + i \sin\phi} = \cos\phi - i \sin\phi,$$

and therefore $x + 1/x = 2\cos\phi$. Thus $2\cos\phi$ is a solution of $y^2 + y - 1 = 0$: the de Moivre trick has given us $2\cos\phi$. We conclude that $2\cos(2\pi/5) = (-1 + \sqrt{5})/2$. Now it is easy to produce a straightedge and compass construction of the regular pentagon. (But constructions were produced by Euclid, and undoubtedly by earlier Greek mathematicians, without the aid of algebra.)

1.3.2 The Heptagon

We begin the analysis much as for the pentagon. The seventh roots of unity are the roots of $x^7 = 1$. We have the factorization

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1).$$

To find the roots of $x^6 + x^5 + \cdots + x + 1 = 0$, observe this is a reciprocal equation, divide through by x^3, and make the substitution $y = x + 1/x$. We arrive at $y^3 + y^2 - 2y - 1 = 0$. By using the same argument as in the pentagon section, we can show that $2\cos(2\pi/7)$ is a root of this equation. To find the roots of $y^3 + y^2 - 2y - 1 = 0$, eliminate the y^2 term as usual and use Cardano's Formula. It turns out that we are dealing with an irreducible case cubic, but we can get expressions for the roots using cube roots of complex numbers.

[4] A word or sentence that has the same property is called a *palindrome*. For example, "Able was I ere I saw Elba" is attributed to Napoleon. Clever, particularly for someone who didn't speak English.

It is not difficult to show that our cubic in y has no rational solutions, so by Wantzel's Theorem $2\cos(2\pi/7)$ is not straightedge and compass constructible, and therefore the regular heptagon also isn't.[5] This is a special case of the Gauss-Wantzel Theorem discussed in the trigonometric solution chapter.

1.4 Solving $x^n = 1$ by Radicals

The problem is: can we write down the solutions of $x^n = 1$ by using only operations of arithmetic and k-th roots for various k? As we have carelessly phrased it, the question seems to have a trivial answer: use the "formula" $\sqrt[n]{1}$. So let's specify that in any such formula, we have no control over which k-th root we get. In particular, the expression $\sqrt[n]{1}$ might give us the useless number 1.

The problem can be reduced to the case when n is prime, so look at $x^p - 1 = 0$. In the sections on the pentagon and heptagon, we saw that the solutions of $x^p - 1 = 0$ can be expressed by radicals when $p = 5$ and $p = 7$. The technique we used seems to break down for $p = 11$, because putting $y = x + 1/x$ yields a quintic, and we don't know how to handle quintics. A breakthrough came in 1771, when Vandermonde, using a new idea, surprised people[6] by showing that $x^{11} - 1 = 0$ can be solved by radicals.

The next step came thirty years later, when Gauss showed how to solve $x^p - 1 = 0$ by using only the usual arithmetic operations and q-th roots, where q ranges over the prime divisors of $p - 1$. In particular, let p be a prime of the form $2^m + 1$. It follows from the analysis of Gauss that the p-th roots of unity can be expressed using the arithmetical operations and square root. Since these can be carried out with straightedge and compass, it follows that if p has shape $2^m + 1$ then the regular p-gon is constructible.

The analysis of Gauss depended on subtle number-theoretic ideas, and most of cyclotomy belongs more to number theory than to algebra. Some 200 years after Gauss, cyclotomy is still an active research subject.

2 Beyond the Quartic

Many mathematicians tried to find "formulas" for the roots of equations of degree greater than 4. Among mathematicians making a serious attempt, we

[5] By allowing other construction tools, we *can* construct the regular heptagon. For example we can do it with compass and *marked straightedge*, that is, a ruler with two marks on its edge. We can also do it with compass and a couple of carpenter's squares.

[6] Lagrange had reduced the problem to solving a quintic, but then given up.

have Gregory, Leibniz, and Tschirnhaus in the late seventeenth century, and Bezout and Euler in the eighteenth. After many failures, opinion began to shift, and workers began to suspect that such a "formula" cannot be found. Already by 1771 Lagrange expresses doubt. Some thirty years later, Gauss is also doubtful. By then, Ruffini had given a flawed but fundamentally correct (this is controversial) argument that the general quintic cannot be solved "by radicals." When they were very young, both Abel and Galois, who would settle the problem once and for all, produced what they thought was a general solution. They soon changed their minds.

2.1 Symmetric Functions of the Roots

Cardano was aware, at least in the cases where he knew all the roots of the cubic $x^3 + ax^2 + bx + c = 0$, that their sum is $-a$. Some 50 years later, Viète knew that if the roots are r_1, r_2, and r_3 then $r_1r_2 + r_2r_3 + r_3r_1 = b$ and $r_1r_2r_3 = -c$. A generalization to the degree n polynomial is stated by Girard in 1629, by Descartes 8 years later. None of them has a clear idea of what kind of objects these "roots" might be: their calculations are purely formal.

Definition 1. A polynomial $P(x_1, x_2, \ldots, x_n)$ is called *symmetric* if it is left unchanged by any permutation of the variables x_1, x_2, $\ldots$, x_n.

Let σ_1 be the sum of all products of the x_i taken one at a time (that is, $x_1 + x_2 + \cdots + x_n$, or $\sum_{1 \le i \le n} x_i$). Let σ_2 be the sum of the products of the x_i taken two at a time, that is, $\sum_{1 \le i < j \le n} x_i x_j$ and so on up to σ_n, the sum of the products of the x_i taken n at a time, that is, $x_1 x_2 \cdots x_n$. The σ_i are called the elementary symmetric polynomials in n variables.

Newton studied symmetric polynomials in the mid 1660s, showing that any symmetric polynomial $Q(x_1, \ldots, x_n)$ can be expressed as $R(\sigma_1, \ldots, \sigma_n)$ where R is a polynomial, and that moreover if Q has integer coefficients then so does R. So in particular if r_1, r_2, $\ldots$, r_n are the roots of $P(x) = 0$ then $Q(r_1, \ldots, r_n)$ can be expressed simply in terms of the coefficients of P. From the time of Lagrange and Vandermonde (1770) onward, the study of symmetric functions of the roots became the main tool in the theory of equations.

2.2 Lagrange and Vandermonde

The first major new developments came in 1770–1771, with the publication of Lagrange's *Réflexions sur la résolution algébrique des équations* and

Vandermonde's much less famous *Mémoire sur la résolution des équations.*[7] Also appearing in 1770 was Waring's *Meditationes Algebraicae,* which is less deep, but has themes in common with the work of Lagrange and Vandermonde. And in 1770 there was also an interesting paper on the quintic by Malfatti.[8]

It is not possible to make a quick summary of Lagrange's *Réflexions*—for one thing it is 217 pages long. Lagrange begins with a study of the quadratic, cubic, and quartic. Of course he knows all of the "tricks" that can be used to solve these equations. But Lagrange wants to know whether the tricks can be generated from deeper structural considerations. He succeds by exploiting systematically certain *symmetries.*

We describe Lagrange's analysis of the cubic in order to indicate the flavour of the work. Let the roots of the cubic (in some order) be x, y, and z (assume these are distinct, it makes no difference). Let $t_1 = x+\omega y+\omega^2 z$, where ω is $(-1+\sqrt{-3})/2$. Imagine permuting x, y, and z in all possible ways. Then t_1 is taken by the 6 possible permutations to t_1, t_2, $\dots$, t_6. It is convenient to let $t_2 = z+\omega x+\omega^2 y = \omega t_1$, $t_3 = y+\omega z+\omega^2 x = \omega^2 t_1$, $t_4 = x+\omega z+\omega^2 y$, $t_5 = \omega t_4$ and $t_6 = \omega^2 t_4$.

Let $f(X) = (X-t_1)(X-t_2)\dots(X-t_6)$. The coefficients of $f(X)$ are symmetric in x, y, and z. After some possibly painful calculation, they can therefore be expressed in terms of the elementary symmetric functions in x, y, and z, and therefore in terms of the coefficients of the cubic. So we can think of the coefficients as "known."

It turns out that $f(X)$ is a *quadratic* in the variable X^3. We could show this by computing, but it is better to *imagine* computing.

The polynomial $(X-t_1)(X-\omega t_1)(X-\omega^2 t_1)$ is just $X^3-t_1^3$, for t_1, ωt_1, and $\omega^2 t_1$ are the three cube roots of t_1^3. Similarly, $(X-t_4)(X-t_5)(X-t_6) = X^3-t_4^3$. We conclude that

$$f(X) = (X^3-t_1^3)(X^3-t_4^3),$$

meaning that $f(X)$ is a quadratic in X^3. (A simpler way of putting the result is that $(u+\omega v+\omega^2 w)^3$ only takes on *two values* as (u,v,w) ranges over the six permutations of the roots.)

[7]Lagrange's name is French, but he wasn't. Vandermonde's name is not French, but he was. Vandermonde is chiefly (but wrongly) remembered for the so-called *Vandermonde determinant.*

[8]The four men worked independently. "Coincidences" of this type are fairly frequent in mathematics, maybe more so than simple probability models would suggest. It seems that sometimes a generation has to pass before the next jump forward. That would come around 1800.

Let t be a solution of this quadratic. The roots of the original cubic can be renamed if necessary so that $t = x + \omega y + \omega^2 z$. It is easy to verify that

$$(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$$

is symmetric in x, y, and z, so it can be simply expressed in terms of the coefficients of the original cubic, and can be considered known. Call the result w. Recall that $x + y + z$ can easily be read off from the cubic, so it is also known.

Thus $x + y + z$, $x + \omega y + \omega^2 z$ (that is, t) and $x + \omega^2 y + \omega z$ (that is, w/t) are known, and now to find x, y, and z only requires solving a system of three linear equations.

A similar symmetry argument works for the quartic, and trivially for the quadratic. Lagrange explores the idea for polynomials of degree bigger than 4, gets many results, but begins to think there may be insuperable difficulties even at the quintic. In the process draws attention to the importance of what would later be called groups of symmetries.

2.3 Ruffini

In 1799, Ruffini's privately published two-volume *Teoria Generale delle Equazioni* appears. It claims to prove that the "general" quintic can't be solved by radicals, that is, by any mixture of basic operations of arithmetic and n-th roots. The work is based on foundations laid by Lagrange.

Mathematicians were almost uniformly skeptical about the validity of the argument, perhaps because of obscurities and gaps, perhaps because of the length of the work, perhaps because Ruffini was an outsider. There is no evidence anyone read the book.

Ruffini returned to the problem time and again over the next dozen years, adding detail, including interesting results on permutation groups. Some of these caught the attention of Cauchy, who went on to do significant work on permutation groups and was the only mathematician of stature to say that Ruffini's proof was basically right. For many years the concensus among historians was that Ruffini's proof was fatally flawed, that Abel was the first to show that the general quintic can't be solved by radicals. But opinion has shifted recently to the view that Ruffini's argument was fundamentally correct.

2.4 Abel (1802–1829)

In 1824, Abel[9] published privately a proof that the general quintic is not solvable by radicals. A version of the paper appeared in the prestigious *Crelle's Journal* in 1826. The proof was generally accepted as correct, though in fact there is a serious, albeit fixable, gap. Just as with Ruffini, the proof rested on the pioneering work of Lagrange.

2.5 Galois (1811–1832)

Galois was killed before turning 21, but his work forever changed the direction of algebra. The work is unfortunately much too deep to describe here.

Ruffini and Abel had shown that the "general" quintic is not solvable by radicals. That means that if we think of the coefficients as parameters, there is no formula in terms of these parameters, using arithmetical operations and n-th root, for the roots of the quintic.[10] But *some* quintics obviously have roots expressible using these operations. Which ones?

Galois associated with any polynomial equation a new kind of structure, now called the *Galois group*[11] of the equation, and proved a theorem that links solvability of an equation by radicals with structural properties of its Galois group. The tools that Galois forged would turn out to be immensely powerful.

There were, inevitably, gaps and obscurities in Galois' work. Its radical newness also slowed acceptance. For 14 years after his death, the work lay more or less unread. In 1846 Liouville published most of Galois' manuscripts in his influential *Journal de Mathématiques.* By the 1860s, the ideas had been generally absorbed by the algebraic community. They continued to be developed in the twentieth century, in ever greater abstraction and generality. Classical algebra—that is, algebra as theory of equations—began

[9]Despite his all too short life, Abel is one of the great mathematicians of the nineteenth century.

[10]In the early 1960s, the Ontario government introduced a "truth in lending" law intended to force banks to disclose the true rate of interest they were charging. Banks were naturally averse to telling the truth, and waged a campaign against the legislation. To find the true rate of interest sometimes requires solving a high degree equation. The banks had an ad claiming that Galois had proved this is impossible! That's a lie. There are, and there were at the time, easy algorithms for computing the true interest rate to absurdly high accuracy.

[11]Groups are essential in describing symmetries; they have found uses in nearly every branch of mathematics. And one cannot do crystallography, or properly describe the elementary particles, without group theory.

with al-Khwārizmī. In the work of Galois, modern structure-based algebra is born.

3 The Fundamental Theorem of Algebra

We sketch the history of this theorem, stripped of all technical detail. Gauss gave the theorem its impressive name. But the result is not all that fundamental. It is also not really a theorem of algebra if algebra is taken in its modern sense.

3.1 Statement of the Theorem

We give three versions of the Fundamental Theorem. It is not hard to get from any of the versions to any other.

Theorem 1. *Let $P(x)$ be a non-zero polynomial with real coefficients. Then the number of* complex *roots of $P(x) = 0$ (if we count roots according to their multiplicity) is equal to the degree of P.*

The hard thing in proving the result is to show that there is always at least one root—once that's done, the rest is an easy induction. We have focused on the number of roots, since it was roots that the early workers cared about. An equivalent and structurally more interesting way of stating the result and at the same time avoiding complex numbers, as many people preferred to, including Gauss, goes as follows:

Theorem 2. *Let $P(x)$ be a polynomial with real coefficients, of degree greater than 0. Then $P(x)$ can be decomposed into a product of real polynomials each of which has degree 1 or 2.*

A seemingly stronger result goes as follows:

Theorem 3. *Let $P(x)$ be a polynomial with complex coefficients, of degree greater than 0. Then $P(x)$ can be decomposed into a product of linear polynomials with complex coefficients.*

3.2 Early Developments

3.2.1 Bombelli

Look at the equation $x^3 = 2 + \sqrt{-121}$, which in a sense was considered by Bombelli, though he didn't think in those terms. Bombelli noticed that

$2+\sqrt{-1}$ is a root of this equation. He had faith that similar equations that arise from an analysis of the *casus irreducibilis* can also be solved by using some sort of "impossible" numbers. But he didn't seem to expect that the "impossible numbers" that solve equations of shape $x^3 = a + b\sqrt{-1}$, where a and b are real, will always be *complex* numbers.

3.2.2 Girard

Alfred Girard (1595?–1632) is often credited with giving the first statement of the Fundamental Theorem (the otherwise unknown Roth had stated it in 1608). Girard's statement looks somewhat like Theorem 1, but he doesn't mention complex numbers. Basically he asserts that a polynomial equation should have "enough" roots, but where these roots might lie is left unclear.

By the middle of the seventeenth century, various people are fooling around with the "roots" of a polynomial equation, and finding results that link various coefficients of the polynomial with expressions involving these supposed roots. Only very gradually does it dawn on them that these roots might all be complex numbers, that we do not need to add more types of "impossible numbers" to the mathematical zoo.

3.2.3 Leibniz

At the beginning of the eighteenth century, the calculus is being energetically developed. In particular, various mathematicians are playing with the integration of $P(x)/Q(x)$, where P and Q are polynomials.

Recall the method of *partial fractions*. To use it, we need to express the denominator $Q(x)$ as a product of polynomials with real coefficients, with each factor either linear or quadratic with no real roots. After going through the partial fractions process and integrating, we end up with a rational function plus natural logarithms plus arctan's (some of these may be missing). The question "Can every $Q(x)$ be factored?" has therefore clear importance.[12]

After these preliminaries, we get to Leibniz. In a 1702 paper, he asks whether Theorem 2 is true—then gives a counterexample! He is interested in integrating $1/(x^4 + a^4)$, and offers the factorization

$$x^4 + a^4 = (x + a\sqrt{\sqrt{-1}})(x - a\sqrt{\sqrt{-1}})(x + a\sqrt{-\sqrt{-1}})(x - a\sqrt{-\sqrt{-1}})$$

[12] In the analysis of complex electrical circuits, it is necessary to integrate messy rational functions, and factoring procedures were often needed. These have mostly been replaced by numerical methods. Also, programs such as *Maple* do a good job of factoring.

(whatever that may mean). Leibniz doesn't realize that in fact the fourth power of $(1+\sqrt{-1})/\sqrt{2}$, if computed mechanically, yields -1. So he thinks that $\sqrt{\sqrt{-1}}$ is a new kind of "impossible" number, and concludes that x^4+a^4 can't be expressed as a product of two real quadratics and that integrating $1/(x^4+a^4)$ is a new kind of problem that he can't solve.

Maybe Leibniz was just having a bad day. If he had applied Descartes' Method for solving the quartic, he should have been quickly led to a factorization. Anyway, factoring is easy by just fooling around—here's a quick way of doing it:

$$x^4 + a^4 = (x^2 + a^2)^2 - 2a^2x^2 = (x^2 + a^2 - \sqrt{2}ax)(x^2 + a^2 - \sqrt{2}ax).$$

Leibniz should have noticed all this—Newton had, some 25 years earlier. The point about the mistake is that it shows very clearly that in 1702 it is reasonable for arguably the greatest mathematician of the time (Newton is by then out of action) to believe that "impossible numbers" beyond the complex numbers are needed for factorization. Though complex numbers had been around in the literature for more than a century, they remained meaningless counters to be manipulated.

3.3 Towards a Proof of the Fundamental Theorem

By 1730, both Cotes and de Moivre had found representations of $x^n \pm a^n$ as a product of linear and/or quadratic polynomials. De Moivre had even given a proof. So complex numbers were adequate for factoring an important class of polynomials. Sometime around 1730, the feeling began to develop that what we now call the Fundamental Theorem might be true.

3.3.1 D' Alembert, Euler, Lagrange

The first "proof" of the Fundamental Theorem was published in 1746 by d'Alembert. The strategy of the proof was a good one, but the tools then available weren't sharp enough. Many years later, d'Alembert's idea would be used to give a nice proof of the Fundamental Theorem.

A few years later, Euler made a serious attempt. His strategy was to show that if all polynomials of degree 2^n can be factored over the reals as a product of linear and/or quadratic polynomials, then so can all polynomials of degree 2^{n+1}. Polynomials of degree 4 are easy to handle by Ferrari's or Descartes' method. Euler spent a fair bit of space discussing polynomials of degree 8, but his conclusion is insufficiently justified, and his work on larger degrees is not much more than an expression of hope. He has a separate adequate argument for degree 6.

Lagrange gave a substantially fuller argument, but his proof was incomplete in two ways. He assumed without proof, indeed without explicit mention, that polynomials *can* be expressed as a product of linear polynomials by adding suitable "imaginary" elements. This is routine if one uses standard techniques of modern algebra, but they were not available to Lagrange. Then he sketched in insufficient detail an argument that such a factorization can be transformed into a factorization over the complex numbers. Lagrange's idea was turned into a rigorous proof almost two centuries later.

3.3.2 Gauss

The first proof of the Fundamental Theorem to be fairly widely acknowledged as satisfactory[13] is in Gauss' 1799 dissertation. Gauss later gave three more proofs, the second and third of which are convincing by the standards of the time.[14] There are later proofs which are purely algebraic apart from the use of the Intermediate Value Theorem. There are also proofs in which the Fundamental Theorem is a trivial consequence of a deeper result in complex analysis such as Liouville's Theorem.

Comment. Several times in this chapter we have noted the presence of serious gaps in published proofs. In the past, it was possible to publish informal sketchy arguments that would never pass muster today. The mathematicians of the first rank seldom made serious mistakes, though their work was often incomplete. But mathematicians of lesser rank sometimes published absurdly wrong results. The refereeing system has more or less taken care of that problem. Now, what lesser rank mathematicians publish is rarely wrong—it is merely uninteresting.

[13] It really isn't satisfactory, depending as it does on a topological fact about polynomials in two variables that is taken as obvious by Gauss but is difficult to prove.

[14] At the time of Gauss, the reals hadn't yet been formally defined, and basic properties of continuous functions hadn't been demonstated rigorously, but that's the only thing the second and third proofs lack. The fourth proof is a streamlined version of the first and has the same gap as the first.