Chapter 1

Rotational Motion

1.1 Rotational Kinematics

Now we're going to start going in circles. Everything we did in the first chapters we're going to convert over to the analogous rotational quantities. All angles must be measured in radians. (check the setting on your calculator!!!) Basically, our conversion equation relating the straight-line stuff to the circle-stuff is

$$s = r\theta \tag{1.1}$$

Also, remember that

$$2\pi \; radians = 180^o$$

Now contemplate the following table:

Table 3: Translation versus Rotation					
Translation	variable	units	Rotation	variable	units
position	x	$meters$	angle	θ	$radians$
velocity	$v = dx/dt$	m/s	angular velocity	$\omega = d\theta/dt$	rad/s
acceleration	$a = d^2x/dt^2$	m/s^2	angular acceleration	$\alpha = d^2\theta/dt^2$	rad/s^2

Anything you can do with the x you can do with θ . The derived equations

also tend to be very similar. For example, take the ballistics equations in the following table.

Table 4: Ballistics, Translation vs. Rotation	
Translation	Rotation
$v = at + v_0$	$\omega = \alpha t + \omega_0$
$x = \frac{1}{2}at^2 + v_0t + x_0$	$\theta = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0$

1.2 Examples

1.2.1 Going in Circles

Example 1: Suppose a disk, initially rotating at 10 rad/s, is accelerating at constant angular acceleration of $4rad/s^2$. (a) How long does it take to turn through six hundred radians? (b)What's the angular velocity at that time?

Solution: (a) Start with constant angular acceleration α. We can integrate this baby.

$$\omega = \int \alpha dt \Rightarrow \omega = \alpha t + \omega_0$$

$$\Rightarrow \theta = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0$$

It's just like one dimensional ballistics! Now, plug in the givens and set θ equal to 600 *radians* and solve.

$$600 = \frac{1}{2} \cdot 4t^2 + 10t \Rightarrow t^2 + 5t - 300 = 0 = (t + 20)(t - 15)$$

$$\Rightarrow t = 15\ s$$

(b) Just plug and chug to do this part.

$$\omega = \alpha t + \omega_0 = 4(15) + 10 = 70\ rad/s$$

Here are some obvious and useful relations that can be found be taking derivatives of $s = r\theta$, with r a fixed distance from the axis of rotation :

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

These equations apply to any rotating object, not just circles or disks. Remember that $a_{rad} = v^2/r = \omega^2 r$. Obviously, for constant r only.

1.3 Moments of Inertia and Rotational Kinetic Energy

The last topic in this chapter concerns rotational kinetic energy, which is rather difficult only because it's challenging to calculate moments of inertia.

Moments of inertia take the place of mass. In terms of the above equations, we have

$$K_{tot} = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} I \omega^2$$

Looking at the above equation, it's pretty obvious that I, the moment of inertia, is given by

$$I = \sum_i m_i r_i^2 \qquad (1.2)$$

In the case of continuous solids, this becomes an integral:

$$I = \int r^2 dm \qquad (1.3)$$

The last equation is kind of obscure, so we'll rewrite it in various dimensions. For one dimensional objects:

$$I = intr^2 \lambda ds$$

where λ is the mass per unit length. For two dimensional objects, an area (double) integral is needed:

$$I = \int r^2 \sigma dA$$

while for three-dimensional objects, a volume (triple) integral in necessary:

$$I = \int r^2 \rho dV$$

Example 1. A bicycle with wheels 40 cm in radius is moving at constant speed of 20 m/s. (A) What is the angular velocity of the wheels? (B) Through how large an angle (in radians) does the wheel turn in five seconds?

Example 2. A car starts from rest and accelerates at 4 m/s2. If the wheels are 40 cm in radius, (A) what is the angular acceleration of one of the wheels? (B) What is the angular velocity of the car when the speed of the car is 20 m/s? (C) Through what angle has the wheel turned in this time?

Example 3. A horizontal grindstone, turning counterclockwise at 40 rad/s, slows to rest in 20 seconds. (A) Find the angular acceleration, assuming it's constant. (B) Find the angle through which it turns. (C) Find the angular velocity after it has passed through 50 radians.

Example 4. A jet turbine cranks up to 50 rev/sec in 20 seconds, with constant angular acceleration. Find the tangential and normal components of acceleration at the tip of the fan blades, which are 0.8 meters long.

Example 5. The Earth travels around the sun once a year at a distance of 150 billion meters. (A) Find the angular velocity of the Earth. (B) Find the tangential velocity. (C) What must the acceleration of gravity be at that distance from the sun?

Solution: (A) The Earth travels 2π radians in one year. Angular velocity is in terms of radians per unit time, so in units of radians per year, we have simply

$$\omega = \frac{2\pi}{1\ year} = 2\pi \frac{rad}{year}$$

That was too easy! Usually, though, we use MKS units, of radians per second:

$$2\pi \frac{rad}{year} \cdot \frac{1\ year}{365\ days} \cdot \frac{1\ day}{24\ hrs} \frac{1\ hour}{3600\ sec} = 1.99 \times 10^{-7}\ rad/sec$$

Example 2. A space station consists of a hub 20 meters in diameter with spokes that radiate outwards to 40 meters at the rim. (A) Find the angular velocity of the station that will result in an apparent gravity (normal acceleration) of one gee at the outer rim. (B) What is the corresponding normal acceleration in the hub? (C) How many revolutions per minute does the station make? (D) What is the tangential speed at the hub, and (E) at the rim?

Example 1. A flywheel turning clockwise at 40 rad/s one meter in radius comes to a stop after 8 seconds of uniform deceleration. (A) What is the angular acceleration? (B) What is the normal acceleration at any time? (C) What is the angular velocity when the wheel has gone through 100 radians? (D) What is the total angle the flywheel travels through during that time?

1.3.1 Moments of Inertia

Example 2: Find the moment of inertia of a uniform rod of length L with linear density λ_0 ,where the axis of rotation is perpendicular to the rod and a distance h from one end.

Solution:

$$I = \int r^2 dm = \int_{-h}^{L-h} x^2 \lambda_0 dx$$

Notice the simple substitution, of $r = x$ in this case, and $dm = \lambda_0 dx$

$$= \frac{1}{3}\lambda_0 x^3 \bigg|_{-h}^{L-h} = \frac{1}{3}\lambda_0 \left((L-h)^3 - (-h)^3\right) = \frac{1}{3}\lambda_0 \left(L^3 - 3L^2 h + 3Lh^2\right)$$

Example 3: Thin rod with non-constant density A thin rod of length L with $\rho = Ax^2 + \rho_0$,x the distance from the axis which is perpendicular to the rod and distance h from one end.

Solution: Here, $dm = (Ax^2 + \rho_0 dx$. Plug and chug, arriving at:

$$I = \int_{-h}^{L-h} Ax^2 + \rho_0 dx = \frac{1}{3}Ax^3 + \rho_0 x|_{-h}^{L-h} = \frac{1}{3}A(L-h)^3 + \rho_0(L-h) - \frac{1}{3}A(-h)^3 + \rho_0(-h) = etc.$$

Example 4: Annulus of Material An annulus of material with constant density σ , axis of rotation through the center.

Solution:

$$I = \int\int r^2 dm = \int_0^{2\pi}\int_{r_1}^{r_2} r^2 \sigma r dr d\theta = 2\pi\sigma r^4 \bigg|_{r_1}^{r_2} = \frac{1}{2}\pi\sigma\left(r_2^4 - r_1^4\right)$$

It is customary to rewrite these expressions in terms of the total mass of the object. In this case, simply integrate the surface mass density over the area, and algebraically fit it into the equation for I:

$$M = \int_0^{2\pi}\int_{r_1}^{r_2} \sigma r dr d\theta = 2\pi\sigma \frac{r^2}{2}\bigg|_{r_1}^{r_2} = \pi\sigma\left({r_2}^2 - {r_1}^2\right)$$

$$\Rightarrow I = \frac{1}{2}M\left(r_1^2 + r_2^2\right)$$

Note that in the previous two equations, the θ integration resulted merely in a multiple of 2π .

Example 5: Find the moment of inertia of a hollow cylinder of material, of height h and radius r, with uniform surface density, axis through the center of the hollow.

Solution: This one is easy. All the material is distance r from the axis, so

$$I = Area \times \sigma \times r^2 = (2\pi r h)\sigma r^2 = 2\pi r^3 h\sigma = Mr^2$$

Example 6: Space Station Use the previous two examples to find the moment of inertia of a space station taking the form of two concentric cylinders of radii a and b, with top and bottom coverings (which are annuli).

If you want a real challenge, compute the moment of inertia of a torus of uniform surface density. For the answer, see the appendix. **Example 7:** Compute the moment of inertia of a square of side L and of constant surface density, where the axis is perpendicular to the square and goes through one corner.

Solution: Don't break a sweat on this one.

$$I = \int_0^A \int_0^B (x^2+y^{2)}\sigma dydx = \int_0^A x^2y + \tfrac{1}{3}y^3|_0^B dx =$$
$$= \int_0^A Bx^2 + \frac{1}{3}B^3dx = Bx^{\frac{3}{3}} + \frac{1}{3}B^3x|_0^A =$$
$$\frac{=1}{3}\sigma AB(A^2+B^{2)} = \frac{1}{3}M(A^2+B^{2)}$$

Finally, there's the parallel axis theorem, which states that if you know the moment of inertia around one axis, you can obtain the moment of inertia around a parallel axis by computing

$$I = Md^2 + I_o$$

Here, d is the distance from the old axis to the new one, and I_o is the old moment of inertia. Gravitational potential energy of a rigid body is just the potential energy of the center of mass. Here's a final example.

Example 8: Suppose a cylinder of radius r on a ramp is attached to a pulley of radius r and moment of inertia I as shown. Use conservation of energy to find the velocity of the cylinder at the bottom of the ramp.

Solution:

$$\Delta KE_{rot} + \Delta KE_{trans} + \Delta P_{grav} = 0$$
$$\frac{1}{2}I_p(\omega_{pf} - \omega_{pi}) + \frac{1}{2}I_c(\omega_{cf} - \omega_{ci}) + \frac{1}{2}m(v_f^2 - v_i^2) + mg(h_f - h_i) = 0$$

Note that $r\omega = v$ for both the pulley and the cylinder. Substituting these expressions, and the info on the problem, obtain

$$\frac{1}{2}m_pv^2 + \frac{1}{4}m_cv^2 + \frac{1}{2}m_cv^2 - mgh = 0$$

$$v = [\frac{2m_cgh}{m_p + \frac{3}{2}m_c}]^{\frac{1}{2}}$$

1.4 Temporary Storage

Chapter 2

Rotational Dynamics

2.1 Torque

Torque, $\vec{\tau}$, is a measure of how much angular acceleration can be obtained from a force. The definition is:

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{2.1}$$

The vector $\vec{r}$ points from some reference point (not necessarily the origin) to the point of application of the force, $\vec{F}$. The cross product delivers the part of the second vector that is perpendicular to the first vector, times the first vector. The magnitude of torque is

$$\tau = |\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta \tag{2.2}$$

where θ is the angle between $\vec{r}$ and $\vec{F}$. $\vec{r}$ is often called the moment arm. Levers are often involved in these problems, however even when no lever is evident in a problem, there is still a mathematical moment arm. Torques can be computed around any point whatsoever, even a point floating way off in space, away from the physical object of interest. In a given problem, however, once the reference point is chosen, it must remain so throughout the problem. The units of torque: Newton-meter. Mathematically, τ is

modeled by the sine function, with the maximum effect occurring when $\theta = 90^o$,since $\sin 90^o = 1$. On the other hand, if we push inwards towards the center of the merry-go-round, or pull outwards, we won't get any effective rotational acceleration. Again, the sine function models this behavior, since $\sin 0^o = \sin 180^o = 0$. That matches common sense. Pushing tangentially to the rim of the merry-go-round will produce the optimal effect (since the $\sin 90^o = 1$, the biggest that sine can be). A final factoid: longer levers, which corresponds to longer r, result in a stronger torque, and more effective angular acceleration. This is the classic advantage of obtaining leverage, useful in wrestling, and in moving the world (Archimedes' famous statement: 'If I had a lever long enough, and a fulcrum in which to place it, I could move the world.')

2.2 Newton's Second Law, with Torques

Newton's second law can be converted to the angular form:

$$I\vec{\alpha} = \sum_{i=1}^{n} \vec{\tau}_i \tag{2.3}$$

where is the angular acceleration, and I is called the moment of inertia. For a point mass moving in a circle around a point, the moment of inertia is given by

$$I = MR^2 \tag{2.4}$$

where M is the mass and R is the distance to the axis of rotation. For hoops, disks, and so forth, there are various moments of inertia, which must be derived with calculus. Problems are solved in these rotational contexts much as they are in the linear case. For static problems, where nothing is moving, the sum of the torques and also, separately, the sum of the forces are both zero. For dynamic problems, the torque law must be used.

2.3 Rotational Work and Kinetic Energy

Energy can be stored in a rotating object just as it can be stored in moving objects. This is the principle of the flywheel, which in some experimental vehicles stores energy in the spinning wheel while the car is at a stop light. And to create this rotational energy, it is necessary to do work. This is defined by

$$W = \int_{\theta_1}^{\theta_2} \tau d\theta \tag{2.5}$$

So applying a torque through a given angle will result in work being done. This rotary work will create rotational kinetic energy, which is defined as

$$K_{rot} = \frac{1}{2} I\omega^2 \tag{2.6}$$

where ω is the angular velocity. The work-energy equation can now be changed to include the effects of rotational kinetic energy:

$$W_{other} = \Delta K + \Delta K_{rot} + \Delta U \tag{2.7}$$

As before, the work on the left-hand side doesn't include work done by conservative forces, which are all contained, effectively, in the potential energies.

2.4 Angular Momentum

An angular momentum, $\vec{L}$, can also be defined for point particles:

$$\vec{L} = \vec{r} \times \vec{p} \tag{2.8}$$

where $\vec{p}$ is the usual linear momentum. All objects, therefore, have angular momentum with respect to some reference point. Angular momentum is absolutely conserved, just like linear momentum, for isolated systems.

For extended bodies, the angular momentum must be found by integration. This works out to be

$$\vec{L} = I\omega\hat{n} \tag{2.9}$$

where $\hat{n}$ is a unit vector pointing in the direction of $\vec{L}$.

2.5 Examples

2.5.1 Torque

Example 1. Wrench. A half-meter long wrench is on a nut, the wrench at an angle of 30 degrees below the horizontal. An ape with mass 50 kg grabs the end and dangles. How much torque is applied to the nut? Use the center of the nut as the reference point.

Solution: Apply the torque equation. The force is just the weight of the monkey, mg, while the angle is $\theta = 60^o$. The angle may appear bigger than that–120^o–but it's the angle between the vector directions, not the one made by the figure of the moment arm and the force direction. The force vector points straight down, while the moment arm vector points 30^o beneath the horizontal, a gap of 60^o (or of 300^o). Of course, the symmetry of the sine function is such that the 120^o angle also always works, so not to worry.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = |\vec{\tau}| = rF\sin\theta = 0.5(50 \cdot 9.8)\sin 60^o = 122.5\ N - m$$

Example 2. Galley Slave. A slave in a galley applies a constant force of 150 Newtons to an oar. The oar rotates through a metal loop that is two meters away, the oar going through the loop and extending on into the water, two more meters in length. (A) When the oar is such that the angle between the oar and the force is 90 degrees, how much torque is exerted? Compute it around the metal loop, and find the force exerted on the water and on the galley. (B) Redo the problem if the oar is only three meters long, with one meter on the inside of the loop

Solution: Since the forces are constant, their sum must be zero, just like the sum of the torques. We have from Newton's second law

$$\sum F_i = 0 = F_{sl} + F_{loop} + F_{water}$$

The forces may be positive or negative, of course. We'll take the slave's force on the oar to be positive, which means the force of the water is also in the positive direction, while the force of the metal ring on the oar is negative. If the torques are computed around the metal loop, then the F_{loop} torque contribution is zero, and so

$$\sum \tau_i = 0 = F_{sl}L_{sl} - F_{water}L_{water}$$

which means that

$$F_{water} = F_{sl}\frac{L_{sl}}{L_{water}} = 150\frac{2}{2} = 150\ N$$

From the force equation we then have If the torques are computed around the metal loop, then the

$$F_{loop} = -\left(F_{sl} + F_{water}\right) = -300\ N$$

The reaction force to this loop force is what drives the galley forward. Notice the advantage of the leverage action: the slave has effectively doubled his applied force. In part (B) the analysis is identical, except now the part of the oar inside the boat is only 1 meter long, while outside it's still two meters long. Intuitively, we expect that not to be quite as effective, and it isn't:

$$F_{water} = F_{sl}\frac{L_{sl}}{L_{water}} = 150\frac{1}{2} = 75\ N$$

$$F_{loop} = -\left(F_{sl} + F_{water}\right) = -(75 + 150) = -225\ N$$

Example 3. Strut 1. A strut of mass 10 kg extends straight out from a wall, attached by a hinge. A wire extends from the end of the strut, and is attached to the wall above the hinge, so that there is a 45 deg angle between the strut and the wire. A 20 kg mass hangs from the end. (A) Find the tension in the wire. (B) Find the x and y components of the force in the hinge.

Solution: SYMBOLS " $L=$ length, T=cable tension, m=10 kg weight, M=20 kg weight, g=gravitational acceleration.
Use $\sum \tau = 0$ along with $\sum F_i = 0$. It's advantageous to compute the torques around the hinge, since that knocks the hinge force out of the torque equation (since its moment arm would be zero). There are four torques. The hinge force gives a torque of zero, since it acts at the point we're calculating torques around. The mass of the strut acts effectively from the center of the strut, so will contribute $-mg(L/2)\sin 90^o$. If the strut weren't uniform, we'd have to integrate, but by symmetry, it's balanced, weight-wise, around the center of the strut. This contribution is negative, since it's trying to turn the strut clockwise around the torque-calculation-point, which is the negative angular direction. Similarly, the 20 kg mass on the end contributes $-mgL\sin 90^o$. Finally, the tension contributes a positive torque, since it's trying to rotate the strut counterclockwise, the positive angular direction.

$$\sum_i \tau_i = -mg\frac{L}{2}\sin 90^o - MgL\sin 90^o + TL\sin 30^o = 0$$

Solve for the tension. Notice that the factor of L cancels out, a common occurrence in these strut problems.

$$-10 \cdot 9.8 \cdot \frac{1}{2} \cdot 1 - 20 \cdot 9.8 \cdot 1 + T \cdot \frac{1}{2} = 0$$

$$\Rightarrow -49 - 196 + \frac{1}{2}T = 0 \Rightarrow T = 490\ N$$

The forces are now easy to calculate.

$$\sum F_i = (F_x, F_y) + (0, -mg) + (0, -Mg) + T(\cos 30^o, \sin 30^o)$$

Rip out the components and solve for F_x and F_y:

$$F_x = T\cos 30^o = 490 \cdot \frac{\sqrt{3}}{2} = 200\sqrt{3}\ N$$

$$F_y = mg + Mg - T\sin 30^o = 10\cdot 9.8 + 20\cdot 9.8 - 490\cdot\frac{1}{2} = 98 + 196 - 245 = 49\ N$$

Example 4. Another Strut. A horizontal strut of length 4 meters is attached to one wall by a hinge is supported by a wire attached to its end and running at a thirty degree angle above the horizontal to another wall. The strut has a mass of 10 kilograms. Find (A) the tension in the wire (B) the force in the hinge.

Solution: Compute torques around the hinge. This will take the hinge force out of the torque equation, which is

$$\sum \tau_i = 0 = -mg\frac{L}{2} + TL\sin 30^o$$

Solve for the tension, T:

$$-10 \cdot 9.8 \cdot 2 + \frac{1}{2} \cdot 4T = 0 \rightarrow T = 98\ N$$

Now for the force, use Newton's second law for forces.

$$\sum F_i = 0 \rightarrow (F_x, F_y) + (0, -mg) + T\left(\cos 30^o, \sin 30^o\right)$$

Rip out the x- and y-components and solve the the components of the hinge force:

$$F_x = -T\cos 30^o = -98 \cdot \frac{\sqrt{3}}{2} = 84.87\ N$$

$$F_y = mg - T\sin 30^o = 10 \cdot 9.8 - \frac{1}{2}98 = 49\ N$$

Nice and easy.

Example 5. Kid on a See-Saw. A 40 kg kid sits on one end of a see-saw, while a 80 kg kid sits on the other side. If the see-saw is 3 meters across and rotates around the middle, where should the 80 kg sit so as to perfectly balance?

Solution: Compute the torques around the middle. Then the torque due to the support is zero, the torque due to the see-saw board is also zero (being balanced), so it's just a matter of balancing the torque due to the kids. Call them kid 1 (40 kg) and kid 2 (80 kg). Since they're on opposite sides, one gives positive torque and the other negative torque. We want

$$F_1 L_1 + F L_2 = 0$$

$$M_1 g L_1 - M_2 g L_2 = 0 \rightarrow 40 \cdot 1.5 - 80 L_2 = 0$$

$$L_2 = \frac{1}{2} \cdot 1.5 = 0.75\ meters$$

A sensible, intuitive answer.

Example 6. Disk. A disk with moment of inertia 50 $kg - m^2$ slows to rest in five seconds from an angular velocity of 20 rad/sec, due to the action of a shoe pressing against the side. What torque was exerted, assuming it to be constant?

Solution: The retarding force, here, is the force of kinetic friction, F_k. The torque is $F_k R \sin\theta$, and we assume here that $\theta = 90^o$, since the shoe is pushed straight in, and simple rubbing, intuitively, creates a tangential acceleration. We have

$$I\alpha = \sum \tau_i = \tau$$

There are two unknowns in this equation, so we need to compute one of them, α, first. We have the info necessary:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{0-20}{5-0} = -4\ rad/s^2$$

Plug in and win:

$$\tau = I\alpha = 50 \cdot (-4) = -200\ N - m$$

Example 7. Ladders. A 50 kg kid climbs a 4 meter ladder that leans to the right against a wall, making a right triangle with two 45 degree angles. Consider the wall frictionless, and the mass of the ladder to be 20 kg. If the coefficient of static friction on the ground is 0.3, when does the ladder begin to slip?

Solution: This is a tough problem, but is easier than it looks. Static is easy: sum of the torques is zero, and the sum of the forces is zero. These two equations do the trick. Let +x be to the right, +y up. Let M=mass of the kid, m= mass of the ladder. $\vec{F}_{kid}$= Force of gravity of the kid. $\vec{F}_{lad}$=gravity force of the ladder. $\vec{N}_f$=normal force of the floor. $vecN_w$=normal force of the wall. $\vec{F}_s$=static friction force of the floor. Now let's write down each of these forces:

$$\vec{F}_{kid} = (0, -Mg)$$

$$\vec{F}_{lad} = (0, -mg)$$

$$\vec{N}_f = (0, N_f)$$

$$\vec{N}_w = (-N_w, 0)$$

$$\vec{F}_k = (\mu_s N_f, 0)$$

The friction force is positive because it's preventing the bottom of the ladder from sliding to the left. Now, we have Newton's second law:

$$\sum F_i = 0$$

$$\Rightarrow \vec{F}_{kid} + \vec{F}_{lad} + \vec{N}_f + \vec{N}_w + \vec{F}_k = 0$$

$$(0, -Mg) + (0, -mg) + (0, N_f) + (-N_w, 0) + (\mu_s N_f, 0) = \vec{0}$$

Rip out the x- and y-components, getting:

$$-N_w + \mu_s N_f = 0$$

$$-Mg - mg + N_f = 0$$

We're almost done–well, 2/3 of the way there, anyway. Notice that we've got two of our unknowns already:

$$N_f = Mg + mg = 50 \cdot 9.8 + 20 \cdot 9.8 = 686\ N$$

$$N_w = \mu_s N_f = 0.3 \cdot 686 = 205.8\ N$$

The last thing we have to figure out is the position of the kid when it starts to slip. Let the origin of coordinates be at the left hand bottom of the latter. Let L be the distance along the ladder of the kid from the origin. Compute the torques around the origin. This gives:

$$\sum \tau_i = 0$$

$\vec{F}_s$ and $\vec{N}_f$ don't contribute, because they have zero moment arm. The mass of the ladder, mass of kid, and normal force off the wall all contribute torques:

$$\tau_{kid} + \tau_{lad} + \tau_{wall} = 0$$

$$-MgL \sin 135^o - mg2 \sin 135^o + N_w 4 \sin 135^o = 0$$

$$L = \frac{4N_w - 2mg}{Mg} = \frac{431.2}{490} = 0.88\ meters$$

Notice that in each case, the angle between the moment arm, pointing along the slope of the ladder, and the direction of the applied force, is 135^o.

Example 8. Ladders Again. Repeat the ladder example, where rather than frictionless, the wall has the same coefficient of static friction as the floor.

Solution: This is not a problem. There is an additional force in the y-direction, and an additional torque. The force of friction in the y-direction is

$$\vec{F}_{sy} = (0, \mu_s N_w)$$

and the additional torque, τ_f,computed around the bottom of the ladder, is

$$\tau_f = \mu_s N_w \sin 45^o$$

:No need to redo all the work–tack these terms onto the ends of the right equations, getting:

$$-N_w + \mu_s N_f = 0$$

$$-Mg - mg + N_f + \mu_s N_w = 0$$

$$-MgL \sin 135^o - mg2 \sin 135^o + N_w 4 \sin 135^o + \mu_s N_w 4 \sin 45^o = 0$$

Now to compute the point at which the ladder starts to slip. The only new hassle is that the x- and y-component equations from Newton's second law must now be solved simultaneously. Solve the x-component equation for N_w and plug into the other:

$$-Mg - mg + N_f + \mu_s^2 N_f = 0$$

$$N_f = \frac{Mg + mg}{1 + \mu_s^2} = \frac{50 \cdot 9.8 + 20 \cdot 9.8}{1 + 0.3^2} = \frac{686}{1.09} = 629.36\ N$$

So the wall friction is helping out, though not much. We also get, by plugging back in,

$$N_w = \mu_s N_f = 0.3 \cdot 629.36 = 188.81\ N$$

Now solve the torque equation for L, noticing that $\sin 45^o = \sin 135^o$:

$$L = \frac{4N_w + 4\mu_s N_w - 2mg}{Mg} = \frac{4 \cdot 188.8 + 4 \cdot 0.3 \cdot 629.4 - 2 \cdot 20 \cdot 9.8}{50 \cdot 9.8} =$$

$$= \frac{1510.6 - 392}{490} = \frac{1118.6}{490} = 2.28\ meters$$

So there is, in fact, considerable advantage to having a wall with friction, but the ladder still slips and the kid still falls down.

Example 9. Merry-Go-Round What constant force must be exerted on a merry-go-round turning twice every second if it is to be stopped in 20 seconds. Assume the force makes a sixty degree angle with respect to an outward radial line, against the direction of motion, and that the merry-go-round has mass of 500 kg and radius two meters, with moment of inertia

$I = 0.5Mr^2$.

Solution: First, find the necessary angular acceleration. Second, use the torque form of Newton's second law. From the statement of the problem, the frequency is $f = 2\ Hz$, so $\omega = 2\pi f = 4\pi$. Assume the merry-go-round is moving in the positive angular direction. From rotational ballistics:

$$\omega = \alpha t + \omega$$

$$\rightarrow 0 = \alpha \cdot 20 + 4\pi \rightarrow \alpha = -\frac{\pi}{5}$$

$$I\alpha = \tau = FL\sin\theta$$

The angle, from the description, is 60 degrees. Solve for F and substitute all the numbers.

$$F = \frac{I\alpha}{L\sin\theta} = \frac{0.5 \cdot 500 \cdot 2^2 \cdot (-\pi/5)}{2 \cdot \sin 60^o} = -367.8\ Newtons$$

Example 10. See-Saw Reprise A see-saw massing 100 kilograms and having length 6 meters is pivoted two meters from the left end. (A) Where must a boy massing 60 kg stand so that the see-saw will balance horizontally? (B) What is the force exerted at the pivot point?

Solution: Follow the same technique as in the previous see-saw problem. Obviously, the kid must stand on the short end of the see-saw. Compute torques around the pivot point (though anywhere else would be okay, too). Then

$$F_{kid}L - F_{board}L_B = 0$$

You might think there are two board torques–one from,effectively, the center of the short side and another from the center of the long side–but it also works to consider the board as a whole, to use the center of mass of the whole board, which of course will be right in the middle of the board. The board term is negative because the center of the board is to the right of the pivot point.So $L_B = 1\ m$, and

$$m_{kid}gL - m_B g L_B = 0 \rightarrow L = \frac{m_B}{m_{kid}}L_B = \frac{100}{60} \cdot 1 = 1.667\ m$$

For part (B), we have to look at the forces.

$$\sum F_i = 0 \rightarrow -m_{kid}g + F_P - m_B g = 0$$

$$F_P = m_{kid}g + m_B g = 60 \cdot 9.8 + 100 \cdot 9.8 = 1,568\ N$$

2.5.2 Rotational Dynamics

Example 11. Ball and Block: Grudge Match. A ball and a block, of identical mass M, are to race down a ramp of height h. If at the bottom they're going at the same speed, what must the coefficient of friction be, in terms of the parameters of the problem? Note: The ball has moment of inertia $(2/5)MR^2$.

Solution: We need to write down the work-energy equation twice. For the ball,

$$W = \Delta K + \Delta K_{rot} + \Delta U_{grav} = (K_f - K_i) + (K_{frot} - K_{irot}) + (U_f - U_i)$$

$$W = 0 = \left(\frac{1}{2}Mv^2 - 0\right) + \left(\frac{1}{2}I\omega^2 - 0\right) + (0 - Mgh)$$

Use

$$I = \frac{2}{5}MR^2$$

together with

$$v = R\omega$$

to get

$$\frac{1}{2}I\omega^2 = \frac{1}{2}\frac{2}{5}MR^2\omega^2 = \frac{1}{5}Mv^2$$

Feed this into the work-energy equation. Get

$$0 = \frac{1}{2}Mv^2 + \frac{1}{5}Mv^2 - Mgh$$

Solve for v:

$$\frac{7}{10}Mv^2 = Mgh \rightarrow v = \sqrt{\frac{10}{7}gh}$$

Now let's do the same thing for the sliding block. The work-energy equation reads

$$W_{other} = \Delta K + \Delta U_{grav}$$

The work done by the friction force is

$$W_{fric} = \vec{F} \cdot \Delta\vec{s} = -\mu_k N \Delta s = -\mu_k Mg \cos\theta h \csc\theta$$

Where did all those terms come from? Well , see Chapter 5. But briefly, because of the slant, $N = mg\cos\theta$, where N is the normal force, of course. The distance along the slant, Δs, can be found in terms of the height h

and the angle θ with simple trigonometry: $h/\Delta s = \sin\theta$. Now stick this in the work-energy equation.

$$-\mu_k \cot\theta Mgh = \frac{1}{2}Mv^2 - Mgh$$

Solve for v:

$$\frac{1}{2}Mv^2 = Mgh - \mu_k \cot\theta Mgh = (1 - \mu_k \cot\theta)\, Mgh$$

$$v = \sqrt{(1 - \mu_k \cot\theta)\, 2gh}$$

Comparing the speed of the block and that of the sphere, we see that they will be equal if

$$(1 - \mu_k \cot\theta)\, 2 = \frac{10}{7}$$

$$\Rightarrow \mu_k = \frac{3}{7}\tan\theta$$

The block reaches the highest speed if μ_k is less than the expression on the right. Notice this isn't necessarily the same as winning the race, since that depends on how the velocity varies with position, but here the accelerations are uniform, so the block will win for sufficiently low kinetic friction.

2.5.3 Angular Momentum

Example 12. Collapsing Star. Suppose a star with radius one million kilometers, spinning on its axis once a month, suddenly collapses to a radius of 20 kilometers. How fast is it spinning, now?

Solution: This is typical in the formation of a neutron star, though usually a large fraction of the mass is blown off at the same time, taking with it some of the angular momentum. By conservation of angular momentum, we have:

$$I_i\omega_i = I_f\omega_f \quad \rightarrow \frac{2}{5}Mr_i^2\omega_i = \frac{2}{5}Mr_f^2$$

Obviously, the 2/5 cancels, and the masses, in this case.The initial angular velocity was given: $2pi\ rad/month$. If we divide both sides of the above equation by 2π, we'll convert angular velocity to frequency, the number of times it turns per month, which is 1 time. So

$$\frac{\omega_f}{2pi} = f_f = f_i\left(\frac{r_i}{r_f}\right)^2 = 1\cdot\left(\frac{10^6}{20}\right)^2 = 2.5\times 10^9\ revolutions/month$$

Since there are about 2.6×10^6 seconds in a month, this means that star, which would fit comfortably inside, say, greater Miami Beach, rotates on its axis about a thousand times per second. Such miillisecond pulsars have indeed been observed out there in deep space.

Example.13. The Skater. A skater starts a spin with arms outstretched, knees bent outward, throwing herself around with angular velocity of 2π radians per second. How much will her angular velocity increase if she straightens, pulling in her arms, stretching them up overhead, and making her body as slender as possible? Assume her total mass is 50 kg, that her arms and legs, outstretched as much as possible, are on average 40 cm from the center of her body and mass a total of 10 kg, and that they are brought in to an average position of 10 cm from the center of her body. Assume also that her trunk is a cylinder. Model her extended arms and legs as a pair of weights on a massless strut, one on each side.

Solution: This problem is harder to write down than to do. We need expressions for the moment of inertia before and after. These are, of course, rather approximate. Let m be the mass of one of the extended weights (i.e. 5 kg) and let M be the mass of the rest of her body (40 kg) . R is the radius of her trunk, which may be taken to be 15 cm. The moment of inertia is:

$$I = I_{cyl} + 2mr^2 = \frac{1}{2}MR^2 + 2mr^2$$

where we have added the moment of inertia of a cylinder with two point masses. Calculate the before and after.

$$I_{bef} = \frac{1}{2} \cdot 40 \cdot 0.15^2 + 2 \cdot 5 \cdot 0.4^2 = 0.45 + 1.6 = 2.05\ kg \cdot m^2$$

$$I_{aft} = 0.45 + 2 \cdot 5 \cdot 0.1^2 = 0.55\, kg \cdot m^2$$

Now, use conservation of angular momentum:

$$L_{bef} = L_{aft}$$

$$I_{bef}\omega_{bef} = I_{aft}\omega_{aft}$$

$$\omega_{aft} = \frac{I_{bef}}{I_{aft}}\omega_{aft} = \frac{2.05}{0.55} \cdot 2\pi$$

$$\omega_{aft} = 3.72\omega_{bef} = 7.44\pi\ rad/sec$$

Crude calculation, but now I understand how those skaters do those fantastic spins. The arms and legs are not that heavy compared to the body, but the r^2 term makes a big difference. Quadrupling the speed of the initial spin may be an overestimate, but probably doubling is not that uncommon.

Chapter 3

Gravitation

3.1 Newton's law of gravity

During the years of the bubonic plague, Newton came up with his famous theory of gravity. This is the only possible force law allowing for elliptical orbits with the sun at one focus of the ellipse:

$$\vec{F} = -\frac{GmM}{r^2}\hat{r} \tag{3.1}$$

$G = 6.6x10^{-11} kg - m^3/s^2$, the gravitation constant

r = the distance between the two objects
m= mass of one object–say the object of interest
M= mass of the other object
$\hat{r}$= a unit vector pointing from central object towards the object of interest.

The negative sign means the force is attractive. It is possible to grasp the law intuitively in terms of concentric spheres and numbers of elementary particle masses, but that won't be taken up here. The gravitation constant, G, is presumably the same throughout all time and space, though there has been speculation that it might change with time.

Close to the earth, the distance from an object to the center of the earth,r, doesn't change much, so the gravity force can be modeled as in previous chapters, with $F = mg$. The acceleration of gravity near the surface of the Earth, g, is given by

$$g = \frac{GM}{R_E^2}$$

where M is the mass of the earth, and R_E is the radius of the Earth. In a similar way, the surface acceleration can be found for any world.

When there are several bodies, it's just a matter of computing all the different forces created.

3.2 Potential Energy and Elementary Orbital Mechanics

The potential energy of gravity is given by

$$U = -\frac{GmM}{r} \tag{3.2}$$

Note that

$$F = \frac{-dU}{dr} = -\frac{mMG}{r^2}$$

as it should. We're now ready to apply this law to a study of satellite, planetary, and rocket motion.

3.2.1 Satellite motion and escape velocity

F or a satellite in circular motion about another body such as the Earth, there are a number of interesting relationships that are easy to derive. Using Newton's laws, we obtain

$$ma_r = -\frac{mv^2}{r} = \frac{-GmM}{r}$$

$$\Rightarrow v_{circ} = \sqrt{\frac{GM}{r}}$$

A common, crazy, and inaccurate way to say this is that the "centrifugal" and gravitational forces balance: what's really happening is that the gravity force, providing a centripetal force, bends the path of the particle, which otherwise would be a straight line.

The **Period** of the satellite is the amount of time it takes to go around the Earth (or other body) once. We can easily find the period T, since

$$v = \frac{d}{T} = \frac{2\pi r}{T}$$

where we've used the distance around a circle, $2\pi r$. Combine this and the equation for velocity in a circular orbit, getting

$$\frac{2\pi r}{T} = \sqrt{\frac{GM}{r}}$$

Solve for T:

$$T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

This is almost a statement of Kepler's Third Law. Squaring it, obtain:

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

This is the classic form of Kepler's third law of planetary motion, except in this case it's specialized to the case of circular motion. To get his law, $r \rightarrow a$, where a is the so- called **semi-major axis**, to be discussed again shortly. In the case of circular motion, the radius is equal to the semi-major axis

Next, let's look at another problem, that of escape velocity. Starting at Cape Canaveral, what velocity would be necessary to escape the Earth's pull of gravity? We use conservation of energy:

$$\Delta K + \Delta P_{grav} = 0$$

$$\left(0 - \frac{1}{2} m v_{esc}^2\right) + \left(0 - \frac{-GmM}{R}\right) = 0$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

Here M and R are the mass and radius of the Earth, respectively. Thus, for the Earth, escape velocity is about $11,000\ m/s$.

3.3 Lunar Trajectories

Computing the energy difference between two circular orbits, combined with the equation for rocket Δv discussed in the chapter on Impulse and Momentum, can give a quick and reasonably accurate estimate of the fuel requirements for effecting the orbital transfer. A single, rather hard-to-derive equation, however, will allow fairly sophisticated calculations right away. The equation is

$$\frac{E}{m} = -\frac{MG}{2a} \tag{3.3}$$

$E =$ the energy of the orbit $m =$ the mass of the spacecraft $M =$ the mass

of the world the spacecraft is orbiting. $a =$ the semi-major axis, which is defined by

$$a = \frac{1}{2}\left(r_p + r_a\right) \tag{3.4}$$

r_p is the closest point of approach, called **perihelion** for solar orbits and **perigee** for Earth orbits. r_a is the furthest point away. The derivation of this equation is fairly difficult, and will be taken without proof, here. To illustrate the technique of calculating space missions, we'll now look at Lunar Trajectories. The next example gives the prototype for all missions. More exact calculations are required for an actual mission, of course.

Example: Lunar Transit Suppose a rocket is traveling in low Earth orbit 250 km above the surface, and you desire to transit to an orbit as far away as the moon. What is the approximate necessary Δv ?

Solution: Again, we use conservation of energy. Basically, we calculate our total energy in Earth orbit, after the burn, then the total energy in the other orbit. We assume the rockets are fired tangentially to the orbit, and that the change in position is negligible during the firing. Our rocket engines will have to provide the necessary difference in energy. This energy would ordinarily be applied in two steps: (1) tangentially to the original orbit, putting our spacecraft into an ellipse with apogee at the desired distance (2) at apogee, when another firing of the engine would be necessary to circularize our orbit.

Here, we're interested in obtaining ball park numbers. Therefore, we'll assume the entire impulse is obtained in one firing of the engines. Below, we work in terms of C=E/m; remember, the mass at the end of the thrust phase will be the same as that in the new orbit.

$$C_0 = E_0/m = \frac{1}{2}v_0^2 - \frac{GM}{r_0}$$

$$\frac{1}{2}\frac{GM}{r_0} - \frac{GM}{r_0} = \frac{-1}{2}\frac{GM}{r_0}$$

C_f is found similarly. Then

$$\Delta C = C_f - C_0 = \frac{GM}{2}\left(\frac{1}{r_0} - \frac{1}{r_f}\right)$$

Note the use of the relationship for circular velocity in the above equation.

$$\frac{1}{2}\left(v_0 + \Delta v\right)^2 - \frac{1}{2}v_0^2 = \Delta C$$

$$\Delta v = \sqrt{2\Delta C + v_0^2} - v_0$$

Plugging in the numbers, we obtain, for transition from a circular orbit at 250 km (don't forget to add the radius of the Earth!) to a circular orbit at 380,000 km,

$$\Delta v = 3,167\ m/s.$$

Now, using the rocket equation, we find that the space shuttle can reach the moon with an approximately one-quarter tank refueling in Earth orbit! To wit:

$$\Delta v = v_{ex} \ln \left(\frac{m_s + 0.25 m_{fuel}}{m_s} \right)$$

$$m_s = 140,000 \; kg \quad m_{fuel} = 720,000 \; kg \quad v_{ex} = 4,500 \; m/s$$

$$\Delta v = 3,720 \; m/s$$

m_s is the mass of the orbiter and payload, together with the mass of the external tank. With a light load, there'd be no problem getting back to Earth, given the moon's weak gravity field.

3.4 Interplanetary Trajectories

We now wish to design a mission to another planet. We don't need to be very precise: we're going to think big, and get ball park numbers that in fact will be extremely good for predicting approximate size of space craft, payload, fuel, and so forth. The numbers are probably good to within plus or minus twenty-five per cent.

Our first target is the red planet, MARS. We use the equations developed in the previous section. Our spacecraft will transit from Earth orbit to Solar orbit, to Mars orbit. We'd like to use off-the-shelf technology, so first we'll take the Space Shuttle, fully refueled in Earth orbit. Subsequently, we'll replace the SSME's with a nuclear reactor, and use hydrogen only as the fuel. Can the space shuttle really make it all the way to Mars? Let's find out.

First, calculate the approximate energy per unit kilogram required to transit from a solar orbit at 150 billion meters (the Earth's distance from the sun) to the orbit of Mars, 226 billion meters. Using the same equations as in the previous section, we find

$$\Delta C = C_M - C_E = \frac{GM_{sun}}{2} \left(\frac{1}{r_M} - \frac{1}{r_E} \right) = 1.5 \times 10^{11} \; m^2/s^2$$

A spacecraft in a solar orbit at Earth's orbital radius has the same velocity as the Earth, 29.8 km/s. Taking this to be v_0, we find

$$\Delta v = \sqrt{2\Delta C + v_0^2} - v_0 = 34,494 - 29,800 = 4,694 \; m/s$$

Now, we're going to return to the Earth escape velocity equation, except this time we want to not only escape the Earth, but also have a "hyperbolic" excess velocity of 4,694 m/s. Alternatively, we could escape Earth's influence with one burn and then have another burn to transit to Mars, but this is inefficient. If you've got your druthers, it's better to get the $\Delta v = 4,694 m/s$ all at once at the outset. Remember $\Delta p = \int F dt$? By going faster at the outset, less time is spent in the region where gravity is strong, so there is less reduction of momentum due to the impulse of gravity. With orbital velocity around the Earth of 7,761 m/s, we find that to get 4,694 m/s excess, we need only

$$\frac{1}{2}v_h^2 - \frac{1}{2}(v_0 + \Delta v)^2 + \left(0 + \frac{GM_E}{r_0}\right) = 0$$

$$\Rightarrow \Delta v = \sqrt{\frac{2GM_E}{r_0} + v_h^2} - v_0 = 11,937 - 7,761 = 4,176\ m/s$$

Now, again assuming a single burn to get to Mars orbit from our parking orbit around Earth–and being sure to point the rocket and fire it when we're going with the Earth, so as to add our velocity to the Earth's– (otherwise we'd hop onto an orbit heading inward toward Venus and Mercury!)–let's get an estimate for the range of a fully refueled Space Shuttle in a 250 km orbit.

$$\Delta V = 4,176\ m/s = v_{ex} \ln\left(\frac{m_0}{m_0 - m_b}\right) \Rightarrow m_b = 0.6 m_0 = 520,000\ kg.$$

So 520,000 kg. of fuel must be burned to get the required Δv . Coming back will require a similar energy change, except our load is lightened. Obtain

$$4,196\ m/s = \Delta v = v_{ex} \ln\left(\frac{m_0 - m_b}{m_0 - m_b - m_r}\right) \Rightarrow m_r = 205,574\ kg$$

The total fuel consumed would therefore be about 726,000 kg, about the same amount as carried in the external tank. Strap on a couple of small boosters in Earth orbit, and there wouldn't be any sweat!

How about using nuclear hydrogen rockets, with a similar configuration? I'll let you work out the details. I calculate you'd need about half a million kilograms of hydrogen, assuming an exhaust velocity of 9,000 m/s. This would essentially allow an extra couple hundred thousand kilos for extra supplies and a Mars lander.

How about travel time? Except for simple ellipses with apogee at the target planet, this is a complicated problem. So for the simplest case, we have

$$\frac{4\pi^2}{T}^2 = \left(\frac{GM_{sun}}{a}\right)^3$$

Here, T is the period (time to go all the way around once), and "a" is the semi-major axis of the ellipse, which is the perihelion plus aphelion divided by 2. For our Mars mission, obtain

$$T = 2\pi\sqrt{\frac{a^3}{GM_{sun}}} = 2\pi\sqrt{\frac{\left(1.5\times 10^{11} + 2.27\times 10^{11}\right)/2)^3}{GM_{sun}}}$$

$$= 44,511,409\ s. = 515\ days = 17\ months$$

Half of this is the time of flight, or about eight and a half months.

3.5 Apparent reduction of weight due to rotation

Example: Suppose you're on a planet with twice the mass of Earth and radius of 3,000 km., and suppose it rotates on its axis once every 300 seconds. What is your apparent weight?

Solution:

$$ma = -\frac{mv^2}{R} = -mg + N$$

$$N = mg - \frac{mv^2}{R} = m\left(g - \frac{v^2}{R}\right) = m(31.92 - 15.23) = 16.6\ m$$

So thanks to the rapid rotation, the apparent weight is only one and a half gees. Without the rotation, we'd be putting up with 3 gees.

3.6 Black Holes

For a given, highly compact mass, how far away do you have to be to be able to escape? We can actually get the correct result by a naive application of conservation of energy. The fastest anything can go is c, the speed of light, so

$$\Delta K + \Delta U_{grav} = 0$$

$$\left(0 - \frac{1}{2}mc^2\right) + \left(\frac{GMm}{R_s} - 0\right) = 0$$

$$R_s = \frac{2MG}{c^2}$$

R_s is called the Schwarzschild radius. For a black hole with the same mass as the sun, this radius is about 3 kilometers.

3.7 Einstein's Theory of Gravity

3.8 Examples

Example 4. A satellite is in circular orbit 30,000 km from the center of the Earth. (A) How fast is it going? (B) How long does it take to complete one orbit?

Solution: This is just like the example give in the corresponding section. For circular obits, we have

$$v_{circ} = \sqrt{\frac{MG}{r}} = \sqrt{\frac{5.98 \times 10^{24} \cdot 6.67 \times 10^{-11}}{30 \times 10^{6}}} =$$

$$= \sqrt{1.33 \times 10^{7}} = 3,646\ m/s$$

To get the period, plug into

$$v = \frac{d}{T} == \frac{2\pi r}{T} \rightarrow T = \frac{2\pi r}{v} = \frac{2\pi 30 \times 10^{6}}{3,646}$$

$$T = 5,170\ sec$$

Example 5. Find the location of geosynchronus orbit, the distance where a satellite will be fixed over one spot on the Earth's surface.

Example 2. A satellite is to circle the Earth once every two hours. (A) How far away from the center of the Earth is it? (B) What's the speed of the satellite?

3.9 Storage

Example: Lunar Transit Suppose a rocket is traveling in low Earth orbit 250 km above the surface, and you desire to transit to an orbit as far away as the moon. What is the approximate necessary Δv ?

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Here, T is the period (time to go all the way around once), and "a" is the

semi-major axis of the ellipse, which is the perihelion plus aphelion divided by 2. For our Mars mission, obtain

$$T = 2\pi\sqrt{\frac{a^3}{GM_{sun}}} = 2\pi\sqrt{\frac{\left(1.5 \times 10^{11} + 2.27 \times 10^{11}\right)/2)^3}{GM_{sun}}}$$

$$= 44,511,409\ s. = 515\ days = 17\ months$$

Half of this is the time of flight, or about eight and a half months.

3.11 Apparent reduction of weight due to rotation

Example: Suppose you're on a planet with twice the mass of Earth and radius of 3,000 km., and suppose it rotates on its axis once every 300 seconds. What is your apparent weight?

Solution:

$$ma = -\frac{mv^2}{R} = -mg + N$$

$$N = mg - \frac{mv^2}{R} = m\left(g - \frac{v^2}{R}\right) = m(31.92 - 15.23) = 16.6\ m$$

So thanks to the rapid rotation, the apparent weight is only one and a half gees. Without the rotation, we'd be putting up with 3 gees.

3.12 Black Holes

For a given, highly compact mass, how far away do you have to be to be able to escape? We can actually get the correct result by a naive application of conservation of energy. The fastest anything can go is c, the speed of light, so

$$\Delta K + \Delta U_{grav} = 0$$

$$\left(0 - \frac{1}{2}mc^2\right) + \left(\frac{GMm}{R_s} - 0\right) = 0$$

$$R_s = \frac{2MG}{c^2}$$

R_s is called the Schwarzschild radius. For a black hole with the same mass as the sun, this radius is about 3 kilometers.