Chapter 1

Electric Forces and Fields

1.1 Coulomb's law

The electric field was found experimentally by Coulomb to have the following form:

$$\mathbf{F} = \frac{kQq\hat{\mathbf{r}}}{r^2} \tag{1.1}$$

where $k = 8.99x10^9 coul^{-2}m^3s^-2$, Q is the source electric charge, q is the electric charge of interest, and r is the distance between the two charges. The unit vector, $\hat{r}$ always points from the source charge to the charge of interest. The electric force can be written down, alternatively, as

$$\mathbf{F} = \frac{kQq\vec{r}}{r^3} \tag{1.2}$$

with the vector $\vec{r}$ pointing from the source charge to the charge of interest. Notice that source charge and charge of interest are always interchangeable– depending on which charge the electric force is being calculated. The charge of interest is always the charge for which the force is being calculated. The other charge experiences an equal and opposite force.

Charge is a primitive concept, like mass. Nobody knows exactly what it is, only what it does, how things having charge react to other charged objects. The most common charge in daily experience is that carried by an electron in electrical circuits. Electric and magnetic forces and fields are always in the background, playing a grand role in making the universe what it is. Electric forces make solid objects feel solid, prevent us from falling through the floor, and are responsible for the vast number of different molecules and compounds that create the diversity all around us.

The electric force is very much like the gravity force, except that electric forces can be attractive and repulsive, while gravity forces are always attractive. The gravity force is considerably weaker than the electric force, however. Small amounts of charge, such as obtained by combing hair or rubbing a balloon against cloth, result in electric forces stronger than the gravity force of the entire Earth!

1.2 The Electric Field

The **Electric Field** can be obtained from Coulomb's law by dividing by one of the charges, q. The idea is that, even in the absence of a second charge, an isolated charge creates a vector field that permeates space. This may or may not be true, however it is a useful tool, since often it is possible to create a relatively fixed distribution of charges which then will affect different particles with different charges. The electric field need only be calculated once; to get the different forces on different particles, it is then sufficient to simply multiply the electric field by the charge, in the given location.

Particle physicists regard forces such as the electric force as arising from an exchange of particles (usually called 'virtual'). This doesn't seem entirely consistent with the concept of a field, but in general no one worries about this.

The electric field $\vec{E}$of a charge Q is defined by

$$\vec{E} = \frac{kQ\hat{r}}{r^2} \tag{1.3}$$

All the pieces are as defined for Coulomb's law. The easiest way to calculate an electric field is to put an imaginary charge of 1 Coulomb at the point of interest, and the calculate the force on it by the given distribution of charges. This is because, with a test charge of exactly 1 Coulomb, the electric field is exactly equal to the Coulomb force, except for the units (Newtons for the force, Newtons/Coulomb for the Electric field). Written out, this would look like

$$\vec{F} = q_{test}\vec{E} = 1 \cdot \vec{E} = \vec{E}$$

So if we know how to calculate a force, we can easily calculate a field in the same way, using the 1 C. test charge.

1.3 Calculating Electric Forces and Fields

1.3.1 Magnitude of a Force between two charged bodies

Simply substitute the numbers into Coulomb's law, while ignoring the unit vector.

1.3.2 Calculating Vector Forces in 1-dimension

For any two given charges, substitute in the values (including minus signs, where the charges are negative) and then choose the unit vector to be either $\hat{x}$ or $-\hat{x}$.

It's best to put a triangle around the charge of interest, to avoid flip-flopping on this. The unit vector always points from the field charge towards the point of interest. Add all quantities.

1.3.3 Vector Forces in two dimensions

Put a triangle around the point of interest, then follow the technique as in one dimension. The difference is the unit vectors will be harder to find. Again, there will be for each pair, the charge of interest and the field charge, a different unit vector. This unit vector $\hat{r}$can most easily be found geometrically, in the following way.

Step 1: Compute $\vec{r} = \vec{p} - \vec{s}$, where $\vec{p}$ points from the origin to the point of interest, and $\vec{s}$ points from the origin to the field point. The vector $\vec{r}$ now points from the field point to the point of interest.

Step 2: Find the magnitude of the vector $\vec{r}$. This can be accomplished with the dot product, $r = (\vec{r} \cdot \vec{r})^{1/2}$.

Step 3: $\hat{r} = \vec{r}/r$

All this can also be accomplished with trigonometry, but the above prescription is much, much easier, and generalizes well to continuous distributions of charges.

Once all the unit vectors are found, substitute them in, and then add up all the vectors forces acting on the charge of interest. Of course, these are added component-wise.

1.3.4 Continuous distributions of charge on a curve

It may be that the source charge is, at least approximately, distributed smoothly along some curve in space. To find the force of this line charge on a particle, perform the following steps:
(1) Write down the general expression for a force between the charge of interest, located at point with position vector $\vec{p}$, and an infinitesimal portion of charge on the curve,dq.
(2) Find any convenient vector parametrization of the curve, $\vec{R}$
(3) Find an infinitesimal displacement vector tangent to this curve, $d\vec{s} = d\vec{R}$
(4) Substitute $dq = \lambda ds$ where λ is the charge density per unit length and ds is the magnitude of $d\vec{s}$.
(5) Find, by subtraction, the displacement vector $\vec{r}$ between the infinitesimal element of charge and the charge of interest, $\vec{r} = \vec{p} - \vec{R}$.
(6) Find the magnitude, r, of the displacement vector.

(7) Find the unit vector $\hat{r}$ corresponding to this displacement vector
(8) Assemble all the parts and integrate the resulting expression over the entire curve

1.4 Ballistics in Electric Fields

Charged particles respond to electric fields, and the motion described by Newton's second law. The basic equation is

$$m\vec{a} = q\vec{E} \tag{1.4}$$

If other forces are present, then they are, of course, simply added to the right-hand side of the equation. In general, this equation is more complicated than ballistics in a constant gravity field, because $\vec{E}$ often varies significantly as a particle moves, while near the Earth's surface, the gravity field is approximately constant. However, with parallel plates, for example, it is possible to set up uniform fields where everything works as it did with simple gravitational ballistics. In one dimension, for example, with a constant electric field, we have

$$ma = Eq \rightarrow a = \frac{Eq}{m}$$

Integrating once gives the velocity as a function of time

$$v = \frac{Eq}{m}t + v_0$$

while integrating the velocity gives the position function

$$x = \frac{1}{2}\frac{Eq}{m}t^2 + v_0 t + x_0$$

The constants v_0 and x_0, of course, are the initial velocity and position, respectively.

In two dimensions, the analysis is similar to two-dimensional ballistics in a gravity field.

1.5 Electric Flux and Gauss's Law

Electric flux may be thought of as similar to a flow of fluid through an area, in this case the fluid being replaced by 'electric-ness'. The definition is

$$\phi_e = \int \vec{E} \cdot d\vec{A} \tag{1.5}$$

The main point in defining electric flux is the fact that it figures into the integral form of Maxwell's equations, to be discussed later. The most important of these is Gauss's law:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{inside} \tag{1.6}$$

The $\oint$ is, of course, a surface integral over a closed surface. An example of a closed surface is a beach ball. Gauss's law is actually equivalent to Coulomb's law, and is only true for a force law that goes like r^{-2}. Otherwise, it is very useful for calculating electric fields for charge distributions with spherical, cylindrical, or plane symmetry. It's important to realize that only the charges *inside* a given surface count. The flux of charges outside the surface penetrate one side and then exit the other, resulting in no net flux. This will be illustrated in the examples.

1.6 Examples

Example 1. Electric and Gravity Forces in a hydrogen atom
This is a straightforward calculation, involving only the magnitudes of the forces. The hydrogen atom is assumed to be the Bohr model, where the electron circles the proton like a tiny planet. Using the tabulated values of the masses of the proton and electron, and their charges, obtain:

$$\frac{F_{elec}}{F_{grav}} = \frac{ke^2/r^2}{GM_p m_e/r^2} = \frac{ke^2}{GM_p m_e} \approx 10^{39}$$

Example 2. Three charges on a line
Suppose a one coulomb charge $Q_a$is located at the origin of the x-axis, while a two coulomb charge Q_b is located at x=3. Find the place where the force on a 1 coulomb charge, Q_c, would be zero.

Solution: Because the forces due to the two particles must oppose each other, the particle must be found in between them. Let x be the coordinate of the point in question. Then $3 - x$ represents the distance from the charge at x=3. Summing the two coulomb forces gives the following equation:

$$F_{ac} + F_{bc} = \frac{kQ_aQ_c\hat{\mathbf{x}}}{x^2} + \frac{kQ_bQ_c(-\hat{\mathbf{x}})}{(3-x)^2} \to \frac{k\hat{\mathbf{x}}}{x^2} + \frac{2k(-\hat{\mathbf{x}})}{(3-x)^2} = 0$$

Rearranging results in a quadratic equation:

$$x^2 + 6x - 9 = 0 \to x = -3 \pm 3\sqrt{2} \to x = 1.23$$

The minus root was rejected, since it would put the charge of interest outside the interval [0,3]– an unphysical result, since the forces couldn't balance there.

Example 3. Three charges in a plane (A) Suppose a 1 coulomb charge is at the point (-2,1). Find the force on the charge of 3 coulombs at (2,4). (B) Same problem, but in addition there is another charge of -1 coulombs at (2,0).

Solution: This is simply a matter of computing the unit vector and distance, then plugging in all the values of the charges and adding the vectors. The best way to proceed is to first find a vector that lies down between the two points. This can best be done by subtraction of one coordinate from another–essentially subtracting one position vector from the other. The magnitude is then r.

$$\vec{r} = (2,4) - (-2,1) = (4,3) \rightarrow r = |\vec{r}| = \sqrt{4^2 + 3^2} = 5 \rightarrow \hat{r} = \frac{\vec{r}}{r} = \left(\frac{4}{5}, \frac{3}{5}\right)$$

These results can now be assembled:

$$\vec{F}_{AB} = \frac{kQ_AQ_B\hat{r}}{r^2} = \frac{3k}{25}\left(\frac{4}{5}, \frac{3}{5}\right)$$

For part B, simply calculate the other vector and add it in to the result of part A.

$$\vec{F}_{CB} = \frac{kQ_CQ_B\hat{r}}{r^2} = \frac{-3k}{16}(0,1)$$

$$\vec{F}_{tot} = \vec{F}_{AB} + \vec{F}_{CB} = \left(\frac{12k}{125}, \frac{9k}{125}\right) + \left(0, \frac{-3k}{16}\right) = etc$$

Example 4. Line of charge in one dimension
A line of charge, with charge density given by $\lambda = \alpha x$, lies between $-L < x < 0$. Calculate the electric field at the point x_0.

Solution: Proceed formally, according to the above sequence of steps.
Step 1:

$$d\vec{E} = \frac{kdq\hat{r}}{r^2}$$

Step 2:

$$\vec{R} = (x,0)$$

Step 3:

$$d\vec{s} = d\vec{R} = (dx,0)$$

Step 4:

$$dq = \lambda ds = \lambda dx$$

Step 5:

$$\vec{r} = \vec{p} - \vec{s} = (x_0,0) - (x,0) = (x_0 - x, 0)$$

Step 6:

$$r = (x_0 - x, 0)$$

Step 7:

$$\hat{r} = \hat{x}$$

Step 8:

$$d\vec{E} = \frac{k\lambda dx\hat{x}}{(x_0 - x)^2} = \frac{k\alpha x dx}{(x_0 - x)^2}\hat{x}$$

Substitute $u = x_0 - x$, which then results in

$$\vec{E} = \int_{-L}^{0} \frac{-k(x_0 - u)du}{u^2} = \int \frac{k}{u} - \frac{kx_0 du}{u^2} = k \ln u + \frac{k}{u} = \left(k \ln(x_0 - x) + \frac{k}{x_0 - x}\right)\Big|_{-L}^{0} =$$

$$= k \ln x_0 - k \ln(x_0 + L) + \frac{k}{x_0} - \frac{k}{x_0 + L} = k \ln\left(\frac{x_0}{x_0 + L}\right) + \frac{kL}{x_0(x_0 + L)}$$

Example 5.An infinite line of constant linear charge density
Suppose a line of charge exists along the x-axis having constant charge per unit length, λ. Find the electric field at the point $(0, y_0)$.

Solution: This requires an integral which can be set up intuitively, however it's better to use the above steps.
Step 1:

$$d\vec{E} = \frac{kdq\hat{r}}{r^2}$$

Step 2:

$$\vec{R} = (x, 0)$$

Step 3:

$$d\vec{s} = d\vec{R} = (dx, 0)$$

Step 4:

$$dq = \lambda ds = \lambda dx$$

Step 5:

$$\vec{r} = \vec{p} - \vec{R} = (0, y_0) - (x, 0) = (-x, y_0)$$

Step 6:

$$r = (\vec{r} \cdot \vec{r})^{1/2} = \left(x^2 + {y_0}^2\right)^{1/2}$$

Step 7:

$$\hat{r} = \frac{\vec{r}}{r} = \frac{(-x, y_0)}{(x^2 + {y_0}^2)^{1/2}}$$

Step 8:

$$\vec{E} = \int_{-\infty}^{\infty} k\lambda dx \frac{(-x, y_0)}{(x^2 + {y_0}^2)^{3/2}}$$

The x-component in the integral is an odd function, hence integrates to zero over the given domain. The y-component can be integrated by trigonometric substitution, resulting in

$$E_y = \frac{2k\lambda}{y_0}$$

The quantity y_0 is arbitrary, so this means that the electric field of the line charge falls off as $1/r$, not $1/r^2$.

Example 6. Charges on a semi-circle, E at the origin.
Find the electric field at the origin due to a semi-circle of radius R with constant linear charge density.

Solution: Follow the same steps as before.
Step 1:

$$d\vec{E} = \frac{kdq\hat{r}}{r^2}$$

Step 2:

$$\vec{R} = (R\cos\theta, R\sin\theta)$$

Step 3:

$$d\vec{s} = d\vec{R} = (-R\sin\theta d\theta, R\cos\theta d\theta)$$

Step 4:

$$dq = \lambda ds = \lambda R d\theta$$

Step 5:

$$\vec{r} = \vec{p} - \vec{R} = (0,0) - (R\cos\theta, R\sin\theta) = (-R\cos\theta, -R\sin\theta)$$

Step 6:

$$r = (\vec{r}\cdot\vec{r})^{1/2} = \left(R^2\cos^2\theta + R^2\sin^2\theta\right)^{1/2} = R$$

Step 7:

$$\hat{r} = \frac{\vec{r}}{r} = (-\cos\theta, -\sin\theta)$$

Step 8:

$$\vec{E} = \int_0^{\pi} k\lambda R d\theta \, (-\cos\theta, \sin\theta)$$

Example 7: Ballistics in Electric Fields, one dimension
The electric field near the surface of the Earth is approximately -150 N/C. (A) How much charge would a 70 kg man have to carry (say, on a special suit) in order to negate his weight? (B) Suppose he carried a charge of lesser magnitude than this, say -3 coulombs. If he could jump 0.5 meters without the suit, how high could he jump with the suit?

Solution: (A) This can be solved with Newton's second law. The forces involved are the gravity field and the electric field. To nullify, the sum of the forces must be zero. Write down this expression and solve for q:

$$ma = -mg + Eq = 0 \rightarrow q = \frac{mg}{E} = \frac{70 \cdot 9.8}{-150} = -4.57 \, C.$$

(B) The ballistics equations for constant acceleration are needed here. We'll use all three, though in fact we could get by with only one, using it twice. The acceleration, of course, is given by $a = -g + Eq/m = -9.8 + (-150)(-3)/70 = -3.37 m/s$. We also need the initial velocity. It would be fairly safe to assume that his initial velocity with the charged suit is the same as when he doesn't have the charged suit. The velocity can be obtained from the usual ballistics equation:

$$v^2 - v_0^2 = 2a(y - y_0) \rightarrow 0 - v_0^2 = 2 \cdot -9.8(0.5 - 0) \rightarrow v-) = \sqrt{9.8} = 3.13 \, m/s$$

We could complete this problem by reusing this equation, but for practice, we'll use the other two. The velocity equation, for the case where there is both a constant electric and gravity field, is given by

$$v = \left(-g + \frac{Eq}{m}\right) t + v_0 \rightarrow -3.37t + 3.13 = 0 \rightarrow t = 0.929 \, s$$

For the height,

$$y = at^2 + v_0 t + y_0 = -\frac{1}{2} \cdot 3.370.929^2 + 3.13(0.929) = 1.45 \, m$$

Example 8. Electric Flux through a rectangle in x-y plane
Find the flux of the electric field (1,2,3) through a rectangle in the x-y plane, bounded by x=0, x=2,y=0,y=5.

Solution: First, we need to find $d\vec{A}$. This is the infinitesimal area element for a typical location on the surface, which is always multiplied by a vector perpendicular to the surface. In this case, intuition suffices: $d\vec{A} = dxdy\hat{z}$. Then the formula for flux gives

$$\int \vec{E} \cdot d\vec{A} = \int_0^5 \int_0^2 (1,2,3) \cdot dxdy\hat{z} = \int_0^5 \int_0^2 3dxdy = 30 \, Nm^2/C$$

Example 9. Flux through rectangle at x=2 and parallel to y-z plane
Suppose a rectangle is parallel to the y-z plane at x=2, with $-1 < y < 2$ and $0 < z < 3$. Find the electric flux of $\vec{E} = (x + y + z, 2, xz)$ through this surface.

Solution: Notice that if the surface isn't closed, an orientation for the surface must be chosen, which corresponds to the area vector pointing up from one side or the other. In this case, we'll choose the positive x-direction for the orientation.

$$\int \vec{E} \cdot d\vec{A} = \int (x + y + z, 2, xz) \cdot \hat{x} dydz = \int_0^3 \int_{-1}^2 x + y + zdydz =$$

$$= \int_0^3 xy + \frac{1}{2}y^2 + yz \,|_{-1}^2 \; dz = \int_0^3 3x + \frac{3}{2} + 3zdz = 3xz + \frac{3}{2}z + \frac{3}{2}z \,|_0^3 =$$

$$= 9x + \frac{9}{2} + \frac{9}{2} = 27 \; N - m^2$$

Notice in the last line we used the fact that x=2 on this rectangle.

Example 10. Gauss's law applied to charges inside a bizarre surface
Suppose charges of 3 C., -6 C. ,7 C, and 9 C. are inside a bizarre shape, while 22 C., -14 C., and 7 C. are outside. Find the total flux through the surface.

Solution: The flux due to the external charges is zero–only the internal charges count. Add them up, getting a total charge of 13 C. The electric flux is subsequently given by Gauss's law:

$$\phi_e = \oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{inside} = 52\pi k$$

Example 11. Gauss's law for a thick spherical conductor with charge at the origin and on the sphere
Suppose a thick spherical conductor has inner radius a and outer radius b. Suppose, furthermore, that +Q coulombs of charge is at the exact center, while -3Q coulombs of charge is placed on the sphere. Find the electric field inside the sphere, inside the conductor, and outside the sphere.

Solution: Inside the sphere, create an imaginary surface a distance r from the central charge Q, with $r < a$. Gauss's law then gives

$$\oint \vec{E} \cdot d\vec{A} = E(4\pi r^2) = 4\pi k Q \rightarrow E = \frac{kQ}{r^2}$$

Inside the conductor the electric field is zero, since the charges are free to move and will cluster around the inner surface, until the surface charge density on the inner surface exactly balanced the charge at the center–equal and opposite it. Then by Gauss's law there will be no electric field inside the metal sphere, so no further charge will collect on the inner surface. Any residual charge will flee, the repulsive effect sending them outward (or rather, displacing large numbers of charges outwards). This migration will continue until all excess charge is on the outer surface. The electric field outside the surface is given by:

$$\oint \vec{E} \cdot d\vec{A} = E(4\pi r^2) = 4\pi k(Q - 3Q) \rightarrow E = \frac{-2kQ}{r^2}$$

The charge distribution will find -1 C of charge on the inner surface of the conductor. This is because negative charge will flow inwards towards the +1 C charge at the center, until such time that the +1 C charge is shielded (i.e. when the inner surface charge amounts to -1 C). And if -1 C is on the inner surface, then -2 C must be on the outer surface, since the total on the thick, spherical conductor is -3 C.

Example 12. Gauss's law for an infinite thin cylinder with surface charge density
Suppose an infinite cylinder of radius R has a constant charge per unit area, σ. Find the electric field inside and out.

Solution: Create an imaginary cylinder inside the cylinder, at radius $r < R$ from the center. No charge is contained inside, so by Gauss's law the electric field must be zero. Outside the cylinder, make a similar imaginary surface, with $r > R$. Then Gauss's law gives:

$$\oint \vec{E} \cdot d\vec{A} = E(2\pi rL) = 4\pi kQ_{inside} = 4\pi k(\sigma 2\pi RL) \rightarrow E = \frac{4\pi k\sigma R}{r}$$

Example 13. Gauss's law for an infinite plane
Suppose an infinite plane carries a uniform charge density σ. Find the electric field. (B) Now suppose there are two metal plates a distance d apart, one of them carrying positive charge and the other carrying an equal and opposite negative charge. Find the electric field between the plates. (This is called a parallel plate capacitor.)

Solution: (A) Make a small pillbox containing a certain area A of charge. Then both end caps each have area A, also. The flux through the sides of the pillbox is zero, since by symmetry the electric field is perpendicular to the plate. Gauss's law gives:

$$\oint \vec{E} \cdot d\vec{A} = E(2A) = 4\pi k(\sigma A) \rightarrow E = 2\pi k\sigma = \frac{\sigma}{2\epsilon_0}$$

(B) In this case, imagine a positive charge placed between the two plates. Assume, for concreteness, that the bottom plate is positive and the top is negative. The positive test charge is repelled upward by the bottom plate and attracted with equal force upward by the negatively charged plate. Thus, the field is doubled, and for a parallel plate capacitor,

$$E = \frac{\sigma}{\epsilon_0}$$

Example 14. Gauss's law for a solid sphere with constant charge density
Suppose a sphere of radius R has constant charge density ρ. Find the electric field inside and outside the sphere.

Solution: Outside the sphere is easy: for $r > R$, Gauss's law gives:

$$\oint \vec{E} \cdot d\vec{A} = E(4\pi r^2) = 4\pi k Q_{tot} \rightarrow E = \frac{kQ}{r^2}$$

This answer is identical to that of a point particle, and in fact this will always be the case for a radially-symmetric charge distribution. For $r < R$:

$$\oint \vec{E} \cdot d\vec{A} = E(4\pi r^2) = 4\pi k Q_{inside} = 4\pi k(\rho \frac{4}{3}\pi r^3) \rightarrow E = \frac{4\pi k \rho r}{3}$$

The total charge is $Q_{tot} = 4\pi r^3 \rho/3$. Solving this for ρ and substituting results in:

$$E = \frac{kQ_{tot}r}{R^3}$$

This is obviously continuous with the external electric field.

Chapter 2

Electric Energy and Potential Fields

2.1 Electric Potential Energy

The **electric potential energy** is similar to gravitational potential energy. It comes from the work done by the electric field on an object moving through the field, and can be defined as follows:

$$W_{ab} = \int_a^b q\vec{E} \cdot d\vec{s} \equiv -(U_b - U_a) = -\Delta U_{ab} \tag{2.1}$$

"W_{ab} is the work done by the electric field as the particle moves from point a to point b. "U_{ab}" is the change in the electric potential energy, also denoted as ΔU. Defined in this way, the electric work can be transferred to the right hand side of the work-energy theorm, which is

$$W_{other} = \Delta K + \Delta U \tag{2.2}$$

"W_{other}" means work due to forces other than electric and gravitational forces. It turns out that the electric potential energy calculated with the above line integral is the same no matter what path is taken from a to b. So the electric field is a conservative field, like the gravity field. The form of U is easy to find for a point particle, which is:

$$U = \frac{kQq}{r} \tag{2.3}$$

The advantage of defining an electric potential energy is that it is a scalar quantity, rather than a vector. Typically, problems involve finding out something at one location, given some information at another location, using the work-energy theorem.

The potential energy of a collection of charges can be found by simply adding up all the contributions of all the different possible pairs of charges. This is equal to the work necessary to bring all the charges together from infinity into the given configuration, as can be seen by inspecting the work-energy theorem. Formally, this can be written down as

$$U = \sum_{i<j} \frac{kQ_iQ_j}{r_{ij}} \tag{2.4}$$

The curious condition $i < j$ is necessary to prevent counting contributions twice, and to avoid counting a non-existent potential energy between a charge and itself.

2.2 Electric Potential

The **electric potential** is very nearly the same as the electric potential energy, and for a point particle is given by

$$V = \frac{kQ}{r} \tag{2.5}$$

As can be seen, the electric potential is obtained from the electric potential energy simply by dividing by q, the charge of interest. In this way, it is possible to discuss fields due to a distribution of particles independent of their affect on a particular charge. Like the electric field, it's convenient to define an electric potential which pervades space and is independent of some particular charge of interest.

The electric potential energy and electric potential are useful in a variety of contexts, and are much easier to use than Coulomb's law, since they are scalars rather than vectors.

2.3 Calculating Potentials

The methods of calculating electric potentials follow the calculations of electric fields, with the difference that no unit vectors need be found, and the answers are all scalars, not vectors. Finally, the potential field of a point particle falls off like r^{-1} rather than r^{-2}.

2.3.1 Point Particles

$$V = \sum_{i} \frac{kQ_i}{|\vec{r} - \vec{r}_i|} \tag{2.6}$$

where $\vec{r}_i$ is a position vector to the i^{th} charge and $\vec{r}$ is a position vector to the point of interest. To find the potential field at an arbitrary point in space, given by the coordinates (x, y, z), the above equation becomes

$$V(x,y,z) = \sum_i \frac{kQ_i}{|\vec{r} - \vec{r_i}|} = \sum_i \frac{kQ_i}{((x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2)^{1/2}}$$

2.4 Examples

Example 1. Two protons, off to infinity
Suppose two protons are placed a distance r_0 away from each other, and subsequently released. What are their velocities at infinity?

Solution: By symmetry, they'll have the same velocity at infinity. Thus conservation of energy gives:

$$\Delta K + \Delta U = (K_f - K_i) + (U_f - U_i) = \left(\frac{1}{2}mv^2 + \frac{1}{2}mv^2 - 0\right) + \left(0 - \frac{ke^2}{r_0}\right) = 0$$

$$v = \sqrt{\frac{ke^2}{mr_0}}$$

Example 2. A proton and an alpha particle
A proton and an α particle (fully ionized helium nucleus) are placed within r_0 of each other and released. Find their velocities at infinity.

Solution: Unlike the previous example, both conservation of energy and of momentum are required. This is because the symmetry of the previous problem has been lost: clearly, the proton and alpha particle will have different velocities at infinity. With an additional unknown, an additional equation is needed. Conservation of energy gives the equation

$$\Delta K + \Delta U = (K_f - K_i) + (U_f - U_i) = \left(\frac{1}{2}mv^2 + \frac{1}{2}MV^2 - 0\right) + \left(0 - \frac{ke(2e)}{r_0}\right) = 0$$

In the above equation, large M and V correspond to the alpha particle, while the lower case m and v refer to the proton. From conservation of momentum:

$$p_{before} = p_{after} \to 0 = mv + MV \to v = -\frac{MV}{m}$$

Substituting this into the conservation of energy equation gives

$$\left(\frac{1}{2}\frac{M^2}{m}V^2 + \frac{1}{2}MV^2 - 0\right) + \left(0 - \frac{ke(2e)}{r_0}\right) = 0$$

$$\left(\frac{M^2 + Mm}{m}\right)V^2 = \frac{2ke^2}{r_0} = 0 \to V = \sqrt{\frac{2mke^2}{M^2 + Mm}}$$

Using the momentum equation, v can also be found.

Example 3. Alpha particle stopped by gold nucleus

An alpha particle (helium nucleus) having kinetic energy of $1 \times 10^25\ J$ flies directly at a gold nucleus (charge 79e). How close does it get?

Solution: Use conservation of energy. The particle is coming effectively from infinity, where the potential is zero.

$$\Delta K + \Delta U = K_f - K_i + U_f - U_i = 0 - K_i + U_f - 0 = -K_i + \frac{k(2e)(79e)}{r_f} = 0$$

$$\rightarrow r_f = \frac{158ke^2}{K_i}$$

Example 4. Two protons and an alpha particle
Two protons and an alpha particle are in an equilateral triangle with sides 10 femtometers in length. If by some mechanism the one proton is released, while the other particles remain fixed, find the velocity of the proton at infinity.

Solution: Assume the alpha particle is on the y-axis, with the protons on the x-axis in the x-y plane. Let the right hand particle escape. Then

$$\Delta K = K_f - K_i = \frac{1}{2}mv^2 - 0 = \frac{1}{2}mv^2$$

where m, v are the mass and velocity of the proton, respectively. Meanwhile:

$$\Delta U = U_f - U_i = \frac{ke(2e)}{r_0} - \left(\frac{ke(2e)}{r_0} + \frac{ke(2e)}{r_0} + \frac{ke^2}{r_0}\right) = -\left(\frac{ke(2e)}{r_0} + \frac{ke^2}{r_0}\right)$$

Note that the term with $e(2e)$, the potential energy of an alpha particle and proton, appears twice in U_i because there are two protons. Assembling:

$$\Delta K + \Delta U = \frac{1}{2}mv^2 - \left(\frac{ke(2e)}{r_0} + \frac{ke^2}{r_0}\right) \rightarrow v = \sqrt{\frac{6e^2}{mr_0}}$$

Example 5. Potential of a line segment of charge.
Find the potential at $x_0 > 0$ for a uniform line of charge lying in $-L < x < 0$.

Solution: This can be done formally, but in this case is very easy, so that formalism will be dispensed with.

$$V = \int_{-L}^{0} \frac{k\lambda dx}{x_0 - x} = -k\lambda \ln(x_0 - x)|_{-L}^{0} =$$

$$= -k\lambda (\ln x_0 - \ln(x_0 + L)) = k\lambda \ln\left(\frac{x_0 + L}{x_0}\right)$$

Example 6. ΔUvia integration
Find change in the electric potential energy for a particle with 3 microcoulombs of charge which is moved from (-1,2) to (3,3) through an electric field $\vec{E} = (0, 20)$.

Solution: The path wasn't specified, so we'll go along two different paths, hopefully getting the same answer (since that answer is supposed to be independent of the path). The easiest path would be to go horizontally and then vertically.

$$\Delta U = -\int_a^b \vec{E}\cdot d\vec{s} = -\int_{-1}^{3}(0,20)\cdot(dx,0) - \int_2^3 (0,20)\cdot(0,dy)$$

The x and y lines were parameterized in the obvious way, $\vec{s} = (x, 2)$ and $\vec{s} = (3, y)$, respectively. The first integral gives zero, while the second integral results in

$$\Delta U = \int_2^3 (0,20)\cdot(0,dy) = \int_2^3 20\,dy = 20\ J.$$

Let's try another path. Parameterize the straight line by first finding the equation of the line, which is

$$y = \frac{1}{4}x + \frac{9}{4}$$

Then

$$\vec{s} = \left(x, \frac{1}{4}x + \frac{9}{4}\right) \rightarrow d\vec{s} = \left(1, \frac{1}{4}\right)$$

$$\Delta U = -\int_{-1}^{3}(0,20)\cdot\left(1, \frac{1}{4}\right) = \int_{-1}^{3} 5\,dx = 5x\,\big|_{-1}^{3} = 20\ J.$$

Example 7. The Electric Field as the Gradient of a potential
Suppose an electric potential is given by $V = x^2 - xy + z^3$. Find the electric field corresponding to this potential.

Solution: This is simply a matter of calculating the three partial derivatives of this expression and assembling them into a vector.

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) = \left(-2x + y, +x, -3z^2\right)$$

Obviously this is a fairly strange electric field, and could only be due to a bizarre charge distribution of some kind, but hey, that's how this calculation is done, and it's easy.

Example 8. Circle of uniform charge in x-z plane.
A circle of charge of radius R with uniform linear density λ lies in the x-z plane, centered at the origin. Find the electric field at the point $(0, y_0, 0)$.

Solution: This is straightforward, following the technique developed for electric fields.
Step 1:

$$dV = \frac{kdq}{r} = \frac{k\lambda ds}{r}$$

Step 2:

$$\vec{s} = (R\cos\theta, 0, R\sin\theta)$$

Step 3:

$$d\vec{s} = (-R\sin\theta d\theta, 0, R\cos\theta d\theta)$$

Step 4:

$$ds = (d\vec{s}\cdot d\vec{s})^{1/2} = Rd\theta$$

Step 5:

$$\vec{r} = \vec{P} - \vec{s} = (-R\cos\theta, y_0, -R\sin\theta)$$

Step 6:

$$r = (d\vec{r}\cdot d\vec{r})^{1/2} = \left(R^2\cos^2\theta + y_0^2 + R^2\sin^2\theta\right)^{1/2} = \left(R^2 + y_0^2\right)^{1/2}$$

Step 7:

$$V = \int_0^{2\pi} \frac{k\lambda R d\theta}{(R^2+y_0^2)^{1/2}} = \frac{2\pi k\lambda R}{(R^2+y_0^2)^{1/2}}$$

Notice in the final integral all quantities were constants, resulting in a very simple calculation!

Example 9. Find a potential difference with differential equations. Suppose the electric field for a given distribution of charges is given by $\vec{E} = (y, x+y^2, 2z)$. Find the potential function, V.
Solution: Start with a set of differential equations, defined by

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) = (y, x+y^2, 2z)$$

This gives three partial differential equations from which we can get V. Start with the x- component, and integrate one component at a time, always remembering to add an unknown function at each step.

$$\frac{\partial V}{\partial x} = -y \rightarrow V = \int -y dx = -xy + f(y,z)$$

$$\frac{\partial V}{\partial y} = -x + \frac{\partial f}{\partial y} = -x - y^2 \rightarrow \frac{\partial f}{\partial y} = -y^2 \rightarrow f = \int -y^2 dy = -\frac{1}{3}y^3 + h(z)$$

So

$$V = -xy - \frac{1}{3}y^3 + h(z)$$

Finally

$$\frac{\partial V}{\partial z} = h'(z) = -2z \rightarrow h = -z^2 + C \rightarrow V = -xy - \frac{1}{3}y^3 - z^2 + C$$

Chapter 3

Capacitors

3.1 Basic Facts

Capacitors store charge by exploiting geometry, positive charges on one surface holding negative charges on another surface through the coulomb force.. They are easy to make. A simple example is the parallel plate capacitor, which consists of two metal plates facing each other and connected by a battery. Positive charge collects on one plate, and negative charge collects on the other. Even if the battery is now removed, the charges remain, and store electrostatic energy. In TV sets and other electronic devices, there can be capacitors with dangerously large charge. Capacitance is defined by

$$C = \frac{Q}{\Delta V} \tag{3.1}$$

As simple as this equation seems, it isn't always easy to calculate the capacitance of a given distribution of charges, with a given geometry. The parallel plate capacitor is easy, however. Let A be the area of the plate, σ the surface charge density, and d the distance between the plates. Then

$$C = \frac{Q}{\Delta V} = \frac{\sigma A}{Ed} = \frac{\sigma A}{\frac{\sigma}{\epsilon_0} d} = \frac{\epsilon_0 A}{d} \tag{3.2}$$

Two capacitors in series have the same charge. This is because of the geometry. Consider two capacitors in series, one to the left of the other. If Q collects on the left hand plate, then -Q must collect on the other plate, drawn by this positive charge. This is actually an isolated conductor that looks like an 'H'. Since the isolated conductor has no net charge, the other, right hand plate of the H must have a charge of +Q. Finally, this means the right hand plate of the second conductor must have charge -Q. This fact can be used to find the single capacitor which is equivalent to two capacitors in series. Let ΔV be the drop across both capacitors. Then

$$\Delta V = \frac{Q}{C_{eq}} = \Delta V_1 + \Delta V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

Note that the fact that the charge is the same on both capacitors and on the equivalent single capacitor has been exploited. From this equation, it is easy to see that for capacitors in series:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{3.3}$$

Capacitors in parallel have the same voltage drop across them. This is easy to understand, since in the circuit, to either side of either capacitor, the electrical environment looks identical. It's analogous to a pair of masses at the same height in a gravity field. The gravity potential, gh, is the same for both of them (though the energy will be different, given different masses). By the geometry, different charges can be stored independently in the different capacitors. These two facts can be exploited.

$$Q_{tot} = Q_1 + Q_2 \rightarrow C_{eq}\Delta V = C_1\Delta V_1 + C_2\Delta V_2$$

Since all the voltages in the above equation are the same, we have, for capacitors in parallel,

$$C_{eq} = C_1 + C_2 \tag{3.4}$$

The stored energy on a capacitor can be easily found.

$$\Delta V = q/C \rightarrow dU = dq\Delta V = dq\, q/C \rightarrow U = \int_0^Q \frac{q}{C} dq$$

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\Delta V^2 \tag{3.5}$$

3.2 Dielectrics

Dielectric materials (aka **dielectrics** are composed of molecules that have some dipole moment associated with them, or higher moment. When an electric field is applied, the molecules try to align themselves with the field, with the more positively charged regions trying to move in the direction of the electric field, and the negatively charged parts trying to go the opposite way. In a capacitor, from the illustration, it is clear that a simple polar molecule will try to arrange so that its negative pole is closer to the positive plate, while the positive pole is closer to the negative plate. This will effectively neutralize, partly, the charges on the plates, allowing more charge to be stored. Putting dielectrics inside a parallel plate capacitor (or any other kind of capacitor) increases the capacitance. Dielectrics also have an added benefit of preventing collapse of the plates, which would attract each other, being oppositely charged.

The dielectric-ness of a material is modeled by a dielectric constant, K, which is equal to one for free space but larger for various materials. The effect of the dielectric material is effectively to reduce electric fields and potential differences, since the positive charges in the material bend closer to the negative plate in the capacitor, while the negative charges in the material bend towards the positive plate. The new potential difference will be given by

$$\Delta V_{new} = \frac{\Delta V_{old}}{K} \tag{3.6}$$

with a similar equation holding for the electric field. This is for a fixed original field, in terms of fixed charges. When a battery enters the system, there is the possibility of flowing charge, which prevents any change in potential. This will be illustrated in the example, below.

Alternatively, the dielectric constant is combined with ϵ_0, the permittivity of free space, to define the something called the **permittivity** ϵ in a substance:

$$\epsilon = K\epsilon_0 \tag{3.7}$$

When electric field calculations are carried out solely in a region having permittivity ϵ, then using this ϵ instead of ϵ_0 will take care of the dielectric effect. When there are several regions, each with a different dielectric material, there will be discontinuities in the electric field at the boundaries, though the potential, due to the arbitrary constant, can still be taken as continuous.

3.3 Examples

Example 1: A parallel-plate capacitor, with everything on it
A parallel-plate capacitor is connected to a 6-volt battery. If $A = 0.03\ m^2$, and the separation is $d = 2\ mm$. (A) Find the capacitance. (B) How much charge is on the plates? (C) Find the surface charge density on each plate. (D) Find the electric field between the plates. (E) After disconnecting the battery, a dielectric with K=3 is inserted in the gap, completely filling it. Find the voltage drop and the electric field between the plates. (F) Reconnect the battery, keeping the dielectric material in between. Find the voltage drop between the plates, and the electric field. (G) Find the new capacitance, and from that, the total stored charge. (H) Find the charge density on the plates, and the electric field between the plates by using the permittivity, (I) Find the energy stored on the plates without the dielectric (J) Find the energy stored in the capacitor when connected to the battery with dielectric inserted. (K) Suppose the battery was disconnected before the dielectric was put it. How much work was done on the dielectric material?

Solution: Straightforward application of the definitions yields all the answers.
(A)

$$C = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 \times .03}{2 \times 10^{-3}} = 15\epsilon_0\ farads$$

Note that since $\epsilon_0 = 8.84 \times 10^{-12}$, this works out to $C = 133 \times 10^{-12}\ Farads$, or 133 $picofarads$.

(B)

$$C = Q/\Delta V \rightarrow Q = C\Delta V = 133 \times 10^{-12} * 6 = 7.96 \times 10^{-10} coulombs.$$

Given the charge on a proton or electron, this is really quite a few charges, equivalent to about five billion excess electrons on the negative plate.
(C)

$$\sigma = Q/A = 90\epsilon/0.03 = 3000\epsilon_0 \ C/m^2$$

(D) From Gauss's law, $E = \sigma/\epsilon_0 = 3000 \ N/C$ (E)

$$\Delta V_{new} = \frac{\Delta V_{old}}{K} = \frac{6}{3} = 2 \ V.$$

(F) This is a no-brainer. The voltage drop across the capacitor must equal the gain across the battery, or 6 volts, as before. The electric field will again be $E = V/d = 3000 \ V$ This seems strange, since there is now a dielectric inserted, but follows from the fact that electric fields are conservative, so that around any loop the gains and drops in potential must equal zero. What is different, however, is the amount of charge carried on the plates, as will be seen. (G)

$$C_{new} = \frac{\epsilon A}{d} = \frac{K\epsilon_0 A}{d} = 3C_{old} = 45\epsilon_0 \rightarrow Q = C_{new}\Delta V = 270\epsilon_0 \ C$$

Notice that the total charge is larger, while the voltage drop is the same. (H)

$$\sigma = Q/A = 9000\epsilon_0 \ C/m^2 \rightarrow E = \frac{\sigma}{\epsilon} = \frac{\sigma}{3\epsilon_0} = 3000 \ N/C$$

So the same answer is obtained as before, using a different method.
(I)

$$U = \frac{1}{2}C\Delta V^2 = \frac{1}{2} \times 133 \ pF \times 6^2 = 4,788 \ pJ$$

(J) With the dielectric inserted, we must use the new capacitance, which is three times the old. Since the capacitor is still connected to the battery, the voltage drop is still 6 volts. Hence the energy stored is 14,364 pJ, where pJ stands for picojoules.
(K) First, we find the new energy. The voltage is reduced by the factor 1/K, while the capacitance is increased by a factor of K. The energy thus is changed by a factor of $K \times 1/K^2 = 1/K$. Since K=3, 2/3 of the electric energy of the capacitor has gone into putting torque on the dielectric material.

Example 2. Collections of Capacitors in a circuit
A 1 microfarad and 2 microfarad capacitor are in parallel, and together in series with another 2 microfarad capacitor and a 10 V. battery. Find the charge and voltage drop for each capacitor. (See the figures at the bottom of the solution. You may need to enlarge your view to make them clearer.)

Solution: This is a rather long problem which, however, consists of numerous easy steps. The method consists of creating a series of reductions, combining

capacitors until only a single capacitor is left in a loop with the battery. The voltage drop across the equivalent capacitor is then the same as the gain of the battery, and it's easy to find the charge, using $C = Q/\Delta V$. Subsequently, work backwards through the intermediate diagrams. There are five factoids that are essential:
1. $C = Q/\Delta V$
2. $C_{eq} = C_1 + C_2$ for capacitors in parallel
3. $1/C_{eq} = 1/C_1 + 1/C_2$ for capacitors in series
4. The charge for capacitors in series is always the same (regardless of the capacitances)
5. The voltage drop across parallel capacitors is always the same.

To solve this capacitor circuit, first combine to two parallel capacitors:

$$C_{eq} = C_1 + C_2 = 1 + 2 = 3\,\mu F$$

Now the equivalent 3-microfarad and the 2 microfarad capacitor are in series, in a loop with a ten-volt battery. Combine:

$$1/C_{eq} = 1/C_1 + 1/C_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \rightarrow C_{eq} = \frac{6}{5}\,\mu F$$

At this point, there is a single equivalent 6/5 microfarad capacitor in a circuit with a ten-volt battery. Find the charge:

$$C = \frac{Q}{\Delta V} \rightarrow Q = C\Delta V = \frac{6}{5} \times 10 = 12\,\mu C$$

This charge is the same as the charge on each of the two capacitors in the previous diagram, so the single 2-microfarad capacitor has a 12 microcoulomb charge, as does the combined 3- microfarad capacitor. We need the voltage drop across the single 2-microfarad capacitor:

$$C = \frac{Q}{\Delta V} \rightarrow \Delta V = \frac{Q}{C} = \frac{12\,\mu C}{2\,\mu F} = 6\,volts$$

This means that there is a four volt drop across the equivalent 3-microfarad capacitor, hence a 4- volt drop over each of the capacitors in parallel. Finding the charges on the parallel 1-microfarad and 2-microfarad capacitors is easy:

$$C = \frac{Q}{\Delta V} \rightarrow Q = C\Delta V = 1 \times 4 = 4\,\mu C$$

$$C = \frac{Q}{\Delta V} \rightarrow Q = C\Delta V = 2 \times 4 = 8\,\mu C$$

Example 3: Capacitance of a spherical capacitor Find the capacitance of two spherical shells, of radius r_0 and r, with $r > r_0$.

Solution: This is straight-forward, using the definitions. We have

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{kQ}{r_0} - \frac{kQ}{r}} = \frac{r_0 r}{k(r - r_0)}$$

Example 4. An RC circuit

A 4 ohm resistor is in series with a 5 microfarad capacitor, which has been charged with a 10 volt battery. The battery is removed, and a simple circuit with the resistor and capacitor remains. (A) How long does it take for the charge to drop to half its original value? (B) What is the current at that time? (C) The voltage drop across the resistor? (D) The power dissipated by the resistor?

Solution: This is straightforward, involving a lot of plug-n-chug.
(A)

$$Q = Q_0 e^{-t/RC} \rightarrow \frac{1}{2}Q_0 = Q_0 e^{-t/RC} \rightarrow t = -RCln(1/2) \rightarrow t = 1.38 \times 10^{-5}\ s$$

(B)

$$I = \frac{dQ}{dt} = -\frac{Q_0}{RC}e^{-t/RC}$$

The original charge, Q_0, is given by $C = Q/\Delta V \rightarrow Q = C\Delta V = 50\ microcoulombs$. Then:

$$I_0 = -\frac{Q_0}{RC} = -2.5\ A \rightarrow I = -2.5e^{-1.38\times10^{-5}/2\times10^{-5}} = 1.25A.$$

(C)

$$\Delta V = IR = 1.25 \cdot 4 = 5\ V.$$

(D)

$$P = I^2 R = \frac{\Delta V^2}{R} = 6.3\ watts$$

Example 5. An RC circuit with battery
Same problem as in the previous example, except now a battery with EMF $\mathcal{E}$ is in the circuit.
Solution: As before, the voltage drops around any closed loop must add up to zero. This means that

$$\mathcal{E} - IR - \frac{Q}{C} = 0$$

Substituting $I = dQ/dt$ and rearranging algebraically:

$$\frac{dQ}{dt} = \frac{\mathcal{E}}{R} - \frac{Q}{RC} = -\frac{1}{RC}(Q - \mathcal{E}C)$$

Dividing by $- - \mathcal{E}C$ and 'multiplying' by dt:

$$\frac{dQ}{Q - \mathcal{E}C} = -\frac{dt}{RC}$$

The equation is now separated. Integrating gives:

$$\ln(Q - \mathcal{E}C) = -\frac{t}{RC} + const \rightarrow Q = \mathcal{E}C + Le^{-t/RC}$$

Applying the condition that, at t=0, the capacitor is uncharge (Q=0) results in $L = -\mathcal{E}C$. The result is

$$Q = \mathcal{E}C\left(1 - e^{-t/RC}\right) \tag{3.8}$$

As time progresses, the capacitor approaches full charge, at which time current ceases, as can be determined by taking the derivative of the above equation.

Example 6. Parallel branches in an RC circuit Suppose a battery with resistor R_1 is in parallel with a capacitor and another resistor, R_3, and that initially a switch is open, so that there is no current in any circuit and the charge in the capacitor is zero. Derive expressions for the currents in each of the three branches, once the switch is thrown.

Solution: There are three branches, each carrying a current. Using Kirchoff's laws, we can generate three differential equations for the three unknown currents, which with algebraic manipulation can be massaged into a differential equation in terms of only one current. At either node, the currents give the equation
Eqn. A

$$I_1 = I_2 + I_3 \rightarrow \frac{dQ_1}{dt} = \frac{dQ_2}{dt} + \frac{dQ_3}{dt}$$

The grand loop around the outside gives
Eqn.B

$$\mathcal{E} - R_1\frac{dQ_1}{dt} - R_3\frac{dQ_3}{dt} = 0$$

Finally, the lower loop gives
Eqn.C

$$\mathcal{E} - R_1\frac{dQ_1}{dt} - \frac{Q_2}{C} = 0$$

Solve the first equation for dQ_3/dt and substitute that into the second equation, eliminating dQ_3/dt:

$$R_1\frac{dQ_1}{dt} + R_3\left(\frac{dQ_1}{dt} - \frac{dQ_2}{dt}\right) = 0$$

Solve the equation C for $R_1 dQ_1/dt$ and insert that into the last equation. The two $\mathcal{E}$ terms cancel, yielding

$$-\frac{Q_2}{C} + R_3\left(-\frac{Q_2}{R_1C} + \frac{\mathcal{E}}{R_1} - \frac{dQ_2}{dt}\right) = 0$$

The equation is now expressed solely in terms of Q_2. Rearrange:

$$\frac{dQ_2}{Q_2 - \frac{\mathcal{E}CR_3}{R_1+R_3}} = -\frac{1}{C}\left(\frac{R_1 + R_3}{R_1R_3}\right)dt$$

Integrate:

$$\left|Q_2 - \frac{\mathcal{E}CR_3}{R_1 + R_3}\right| = -\frac{1}{C}\left(\frac{R_1 + R_3}{R_1R_3}\right)t + const$$

$$Q_2 = \frac{\mathcal{E}CR_3}{R_1 + R_3} + \gamma \exp\left(-\frac{1}{C}\left(\frac{R_1 + R_3}{R_1 R_3}\right) t\right)$$

The constant of integration γ can now be determined by the intial condition, $Q_2(0) = 0$.

$$Q_2(0) = 0 = \frac{\mathcal{E}CR_3}{R_1 + R_3} + \gamma \rightarrow \gamma = -\frac{\mathcal{E}CR_3}{R_1 + R_3}$$

Assembling:

$$Q_2 = \mathcal{E}C\frac{R_3}{R_1 + R_3}\left(1 - e^{-\frac{t}{C}\left(\frac{R_1 + R_3}{R_1 R_3}\right)}\right)$$

The currents can easily be found by back substitution.

Chapter 4

Currents and Resistors

4.1 Current, Resistivity, and Drift Velocity

Current is defined as the time-rate of flow of charge, denoted by I, with units in coulombs per second, aka amperes. In terms of derivatives,

$$I = \frac{dQ}{dt} \tag{4.1}$$

If the flow is steady, it's obviously sufficient to divide the amount of charge passing in a certain time by that time, or $I = \Delta Q/\Delta t$. Current is carried by negative charge carriers, called electrons. In general, however, we talk about positive current, which effectively goes the other way. This is because back when the concepts were invented, the electron was unknown. However, negative current in one direction is equivalent to positive in the other, so there is no particular problem with this definition.

The current density is defined as follows:

$$J = \frac{dI}{dA} \tag{4.2}$$

Again, in most simple instances it is sufficient simply to divide the current by the cross-sectional area of the current-carrying conductor. It is also possible to define a **drift velocity**, v_d:

$$J = nqv_d \tag{4.3}$$

In the above equation, n is the number density of charge carriers (typically electrons), and q is the charge on each charge carrier. The number density is related to the physical density and molar density of the substance, and to the number of charge carriers per atom. These charge carriers, also called **valence electrons**, are free to move from on nuclei to another in response to electric fields. The drift velocity is typically no more than centimeters per second or even less. Despite this, electric appliances turn on essentially immediately, since the signal quickly propagates from electron to electron, at nearly the speed of light, when a potential is brought on line. It's a little bit like 'Newton's Cradle', where balls on strings are lined up touching each other, where striking the ball on one end will immediately result in a ball jumping off the opposite end.

The **resistance** R of a substance depends on its length, its cross-sectional area, and on what type of substance it is. The longer the wire made of a given substance, the larger the resistance, while the larger the cross-sectional area, the smaller the resistance. The equation is

$$R = \frac{\rho_e L}{A} \tag{4.4}$$

4.2 Electric Circuits and Kirchoff's laws

Solving circuits with various combinations of resistors is a basic and important problem. Several facts are needed:

1. The sum of voltage gains and losses around any closed loop is always zero.
2. The sum of currents going into or out of a certain location is always zero (counting incoming current as positive and outgoing as negative).
3. For series resistors,

$$R_{eq} = R_1 + R_2$$

4. For parallel resistors

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

The first two facts are also known as Kirchoff's laws. The third fact, the single resistance equivalent to a series of two resistors, follows from the fact that the current is the same in each resistor. The fourth fact follows from parallel resistors having the same voltage drop across them.

Solving a circuit usually means finding the currents in the various different branches. This entails a certain number of linear equations with the same number of unknowns. The general procedure is:

1. Reduce the complexity of the circuit by first applying facts 3 and 4.
2. For n unknown currents, generate n equations using facts 1 and 2.
3. Solve the system of equations by substitution, Gaussian elimination, or matrix methods.

Step 2 can be a little tricky, because applying Kirchoff's laws may result in a dependent set of equations–where one of the equations can be obtained from the others, algebraically. This will become apparent when grinding out the answer results in equations like $0 = 0$. If this happens, just go back and generate a new equation, getting rid of one of the old ones. It's pretty rare that this kind of bad luck happens more than once in a given problem.

4.3 Examples

Example 1: Drift Velocity A wire carries 0.5 amperes of current. If the cross-sectional area is $0.0003m^2$ and the number of charge carriers per meter cubed is $2.25x10^25$, find the drift velocity.

Solution: This is a matter of plugging in the numbers. Since $J = I/A$, the current density is $J = 1,667.67A/m^2$. Then

$$J = 1,667.67 = nev_d = 2.25x10^25 * 1.6x10^{-19} * v_d \rightarrow v_d =$$

$$= 4.63x10^{-4}m/s$$

Example 2: Resistivity
Suppose a wire is melted down and recast so as to have one-half its original length. Find the new resistance in terms of the old.

Solution: The new length, L_n, is such that $L_n = 0.5L_o$, where L_o is the original length. The wire is a cylinder, and the old and new cylinders have the same volume.

$$V_o = A_oL_o = A_nL_n = V_n \rightarrow A_n = 2A_o$$

Plugging these two values into the resistivity equation yields:

$$R_n = \frac{\rho_e L_n}{A_n} = \frac{\rho_e \frac{1}{2} L_o}{2A_o} = \frac{1}{4}\frac{\rho_e L_o}{A_o} = \frac{1}{4}R_o$$

Example 3: Resistors in Series and Parallel
Three resistors, 1 ohm, 2 ohm, and 3 ohms, are in parallel. These are in series with a 4 ohm resistor. Find the equivalent resistance.

Solution:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \rightarrow R_{eq} = \frac{6}{11} Ohms.$$

This is in series with the four Ohm resistor, which can be added, resulting in a total equivalent resistance of $\frac{50}{11}$ Ohms.

Example 4: Three branches and two batteries
(A) Find the currents in the given circuit diagram. (B) What is the power radiated by the 2 Ohm resistor?

Solution: All current directions are right to left and named after the resistors they go through. Note that the direction of the current I_1 has been deliberately chosen to be in the wrong direction, to illustrate it doesn't matter. If you choose the wrong direction, you will end up with a negative sign in the answer. Kirchoff's laws give the following equations:

$$\sum I_i = 0 \rightarrow I_1 + I_2 + I_3 = 0$$

$$\sum \Delta V_i = 0 \rightarrow -2I_2 + 5 + I_1 = 0$$

$$\sum \Delta V_i = 0 \rightarrow -3I_3 + I_1 + 10 = 0$$

Solving the first for I_2 and plugging into the other two equations, after some algebra, gives

$$3I_1 + 2I_3 = -5$$

$$I_1 - 3I_3 = -10$$

Solving results in $I_3 = 25/11\,A.$, $I_2 = 10/11\,A.$, and $I_1 = -35/11 A..$ The negative sign in I_1 means that we chose the direction of current incorrectly. The magnitude of I_1 is correct but the current goes in the opposite direction.

Chapter 5

Magnetic Forces and Fields

5.1 Magnetic Forces

Charged particles respond to a magnetic force given by

$$\vec{F} = q\left(\vec{v} \times \vec{B}\right) \tag{5.1}$$

Here, q is the charge, $\vec{v}$ is the velocity of the charged particle, and $\vec{B}$ is called the **magnetic field**. It's easy to see that the magnetic force on a straight segment of wire carrying current I ought to be:

$$\vec{F} = I\left(\vec{L} \times \vec{B}\right) \tag{5.2}$$

while in general, the force on an element of current is given by

$$d\vec{F} = I\left(d\vec{s} \times \vec{B}\right) \tag{5.3}$$

The important thing to realize is that magnetic forces can't make charges go faster, since the force is always perpendicular to the direction of motion. Magnetic forces can only turn charges. There are, however, forces between one magnet and another that can result in a change in velocity. Magnets attract bits of metal by inducing slight magnetic fields in them, which then are attracted through the magnetic force of one magnet for another. This kind of magnetic induction can be easily observed with a permanent magnet and some paper clips.

The **Lorentz Force Law** is given by

$$F = q(\vec{E} + \vec{v} \times \vec{B}) \tag{5.4}$$

This is just a combination of the electrostatic and magnetic force laws. The equation is particularly useful in the velocity selector problem, which is key in such devices as the mass spectrometer, which was used to separate different isotopes of uranium during the second world war.

5.2 Ampere's Law

Ampere's law is given by

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{inside} \tag{5.5}$$

The equation says that if you integrate around a closed path, the answer you get is equal to the total current enclosed inside the closed path, times the permeability of free space, μ_0. This is similar to Gauss's law, but involves a line integral, not a surface integral.

Using Ampere's law, it is possible to calculate magnetic fields for a variety of simple symmetries. These symmetries correspond to the cases where the magnetic field B is constant on the chosen path of integration, $d\vec{s}$. The cases that can be studied include:

(1) cylindrical symmetry
(2) Plane symmetry
(3) toriodal symmetry

Since in these and similar cases the magnetic field is constant, it is easy to write down the integral. For example, for cylindrical symmetry, write

$$oint\vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_0 I$$

5.3 Examples

Example 1: Magnetic Force on charged particle
A particle traveling 200,000 meters per second in the x-direction enters a magnetic field $\vec{B} = 0.5\hat{z}$. If it turns in a semi-circle of radius 0.05 $meters$ before hitting a photographic plate, what is the mass-to-charge ratio?

Solution: Use Newton's second law, for the circular motion case where $a = v^2/r$.

$$m\vec{a} = q(\vec{v} \times \vec{B}) \rightarrow -\frac{mv^2}{r} = -qvB\sin\theta = -qvB$$

$$\frac{m}{q} = \frac{rB}{v} = 0.05 \cdot 0.5/200,000 = 1.25 \times 10^{-7}\ kg/coulomb$$

Example 2: Magnetic Force on a straight wire segment
How much current must travel through a straight wire, oriented in the positive x-direction, so that the magnetic force due to the field $\vec{B} = 5\hat{x} + 3\hat{y}$ will be sufficient to cancel the force of gravity? The wire has a linear density of 0.02 kg/ meter.

Solution: This is a straightforward application of the law of forces on a current-carrying conductor.

$$m\vec{a} = \sum F_i = (0, 0, -mg) + I\vec{L} \times B = 0$$

$$\vec{L} \times B = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ L & 0 & 0 \\ 5 & 3 & 0 \end{pmatrix} = \hat{x}(0 \cdot 0 - 0 \cdot 5) + \hat{y}(0 \cdot 3 - L \cdot 0) + \hat{z}(L \cdot 3 - 0 \cdot 5) = 3L\hat{z}$$

Plugging into the first equation and extracting the z-component results in

$$-mg + 3IL = 0 \rightarrow -\lambda Lg + 3IL = 0 \rightarrow I = \frac{\lambda g}{3} = 0.0653\ A.$$

Example 3: Velocity selector
A particle traveling in the positive y-direction passes through a velocity selector with fields given by $\vec{E} = (0, 5000, 0)$ and $\vec{B} = (0.2, 0, 0)$. What's the charge-to-mass ratio, if it exits the electric field and turns through a semicircle, impacting a photographic plate 0.06 m. away?

Solution: This problem is solved in two steps. (1) Find the velocity, using the fact that the particles will only exit the velocity selector if the net force (the Lorentz force) is zero. (2) Use the magnetic force law from the previous example to find m/q.

$$\vec{E} + \vec{v} \times B = (0, 0, 5000) + \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & v & 0 \\ 0.2 & 0 & 0 \end{pmatrix} = (0, 0, 0)$$

$$\rightarrow 5000 - 0.2v = 0 \rightarrow v = 25,000\ m/s$$

Once again, use Newton's second law for the circular motion case where $a = v^2/r$. From the first example:

$$m\vec{a} = q(\vec{v} \times \vec{B}) \rightarrow \frac{m}{q} = \frac{rB}{v} = 0.03 \cdot 0.2/25,000 = 2.4 \times 10^{-7}\ kg/coulomb$$

Example 4: Magnetic Torque
Compute the magnetic torque of $\vec{B} = (1, 2, 3)$ on a set of five rectangular circuits of wire with corners at (0,0,0), (2,0,0), (0,3,4), (2,3,4). The current of 0.5 amperes proceeds along the line from (0,0,0) to (2,0,0) and on around the loops.

Solution: The method is as follows, in four steps. (1) Find a normal vector $\hat{n}$ to the surface, making sure that it is oriented via the right hand rule–right thumb in the direction of the normal, fingers waggling in the general direction of positive current. (2) Write down the area vector, $A\hat{n}$. (3) Get the magnetic moment $\vec{\mu} = NIA\hat{n}$, where the number of loops is N and the current is$I$. (4) Find the torque with $\vec{\tau} = \vec{\mu} \times \vec{B}$. It is evident that this is a rectangle with sides of lengths 2 and 5. The area is therefore 10. To find a vector normal to the surface, simply create vectors along two of the sides and compute the cross product. The unit vectors $\hat{x}$ and $(0, 3/5, 4/5)$ point along the two sides of the rectangle.

$$\hat{x} \times (0, 3/5, 4/5) = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \end{pmatrix} = (0, -0.8, 0.6)$$

Notice that with the fingers going in the direction of the current, the right thumb points in the positive z-direction, consistent with the computed normal vector. If this were not the case, it would be a simple matter of multiplying each component by -1.

$$\vec{\tau} = \vec{\mu} \times \vec{B} = NIA \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & -0.8 & 0.6 \\ 1 & 2 & 3 \end{pmatrix} = 25(-0.12, 0.6, 0.8)$$

Example 5: Ampere's law for a long straight wire
Use Ampere's law to find the magnetic field around a long straight wire carrying current I.

Solution: This is easy. First, draw an imaginary circle around the wire at a distance r. By symmetry, the magnetic field will be constant along this circle, and in the same direction as $d\vec{s}$ at all points. This means the line integral will give simply B times the arclength:

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_o I_{inside} \rightarrow B = \frac{\mu_o I}{2\pi r}$$

Example 6: Ampere's law for a coaxial cable
Find the magnetic field at all points for a coaxial cable consisting of a thin, long wire carrying 5 A. of current on the inside, surrounding by a cylindrical shell of radius R carrying 3 A. of current in the opposite direction.

Solution: For $r < R$, the solution is the same as in the previous example, substituting the value of I. For r¿R:

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi r) = \mu_o I_{inside} = \mu_o(5-3) \rightarrow B = \frac{\mu_o}{\pi r}$$

Example 7: Amps law for a solenoid
Find the magnetic field of a long solenoid, where L is the length and N the number of wraps.

Solution: The imaginary line should be a rectangle with one side going through the center of the solenoid and the others completing the other three sides. If the solenoid is long, the contribution to the line integral of these exterior sides will be negligible compared to the contribution of the side going through the center, where the field is strongest, and where it is also in the same direction as the integration, parallel to $d\vec{s}$. Then:

$$\oint \vec{B} \cdot d\vec{s} = BL = \mu_o I_{inside} = \mu_o IN$$

There is an effective multiplication of I by N, since the wire loops around over and over again, each loop creating more magnetic flux. Finally,

$$B = \mu_o I \frac{N}{L} = \mu_o In$$

Example 8: Biot-Savart calculation for a long current-carrying wire
Find the magnetic field due to a long wire carrying current I.

Solution: The set up is nearly identical to that of a long line of charge. Assume the wire is infinite in length and lying along the z-axis, with current going in the positive z- direction. Then, as before:
Step 1:

$$d\vec{B} = \frac{\mu_o}{4\pi}\frac{Id\vec{s}\times\hat{r}}{r^2} = \frac{\mu_o}{4\pi}\frac{Id\vec{s}\times\vec{r}}{r^3}$$

Step 2:

$$\vec{R} = (0, 0, z)$$

Step 3:

$$d\vec{s} = d\vec{R} = (0, 0, dz)$$

Step 4:

$$\vec{r} = \vec{p} - \vec{R} = (0, y_0, 0) - (0, 0, z) = (0, y_0, -z)$$

Step 5:

$$r = (\vec{r}\cdot\vec{r})^{1/2} = \left({y_0}^2 + z^2\right)^{1/2}$$

Step 6:

$$d\vec{s}\times\vec{r} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & dz \\ 0 & y & -z \end{pmatrix} = (-ydz, 0, 0)$$

Assembling these results into the formula from step 1 results in

$$\vec{B} = \frac{\mu_o I}{4\pi}\int_{-\infty}^{-\infty}\frac{(-ydz, 0, 0)}{(y_o^2 + z^2)^{3/2}}$$

This can readily be integrated using the substitution $z = y_o \tan\theta$, resulting in

$$B_x = -\frac{\mu_o I}{2\pi y_o}\hat{x}$$

By symmetry, it's easy to see that this magnetic field circulates around the wire, and agrees with the result obtained from Ampere's law.

Example 9. A circular loop of current
Find the magnetic field at $(0, y_0, 0)$ due to a circular wire of radius R, centered around the origin in the x-z plane and carrying current I.

Solution: Follow the same steps as before.
Step 1:

$$d\vec{B} = \frac{\mu_o}{4\pi}\frac{Id\vec{s}\times\hat{r}}{r^2} = \frac{\mu_o}{4\pi}\frac{Id\vec{s}\times\vec{r}}{r^3}$$

Step 2:

$$\vec{R} = (R\cos\theta, 0, R\sin\theta)$$

Step 3:

$$d\vec{s} = d\vec{R} = (-R\sin\theta d\theta, 0, R\cos\theta d\theta)$$

Step 4:

$$\vec{r} = \vec{p} - \vec{R} = (0, y_0, 0) - (R\cos\theta, 0, R\sin\theta) = (-R\cos\theta, y_0, -R\sin\theta)$$

Step 5:

$$r = (\vec{r}\cdot\vec{r})^{1/2} = \left(R^2\cos^2\theta + y_0^2 + R^2\sin^2\theta\right)^{1/2} = \left(R^2 + y_0^2\right)^{1/2}$$

Step 6:

$$d\vec{s}\times\vec{r} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ -R\sin\theta d\theta & 0 & R\cos\theta d\theta \\ -R\cos\theta & y_0 & -R\sin\theta \end{pmatrix} = (-y_0\cos\theta, -R, -y_0\sin\theta)Rd\theta$$

Step 7:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{(-y_0\cos\theta, -R, -y_0\sin\theta)Rd\theta}{\left(R^2 + y_0^2\right)^{3/2}}$$

This integral looks very difficult, but in fact is very easy. First of all, the x-component boils down to

$$\int_0^{2\pi} \cos\theta d\theta = \sin\theta\big|_0^{2\pi} = 0$$

The z-component works out similarly. Meanwhile, the y-component is just the integration of a constant! This results in a factor of 2π, so the magnetic field has only a y-component, which is

$$B_y = \frac{-\mu_0 I R^2}{2\left(R^2 + y_0^2\right)^{3/2}}$$

Example 10: Magnetic field at the center of several circular current-carrying wires

Suppose a wire carrying current I comes in along the x-axis from $-\infty$, makes a semicircle of radius 1 meter in the positive y-domain, proceeds along the x-axis to x=2, then makes a quarter circle of radius 2 up to the positive y-axis, along which is proceeds to $+\infty$. Find the magnetic field at the origin. **Solution:**First of all, the contributions to the B-field along all the straight-line segments of wire are zero, because in each case the vectors $d\vec{s}$ and $\hat{r}$ are either parallel or antiparallel, so that $d\vec{s}\times\hat{r} = 0$. (Recall that the cross product involves a factor of $\sin\theta$, and that in these two cases $\theta = 0 \; or \; \pi$.) So it is sufficient just to find the contributions due to the two circular segments. These can be found by inspection of the integral in the previous example. Notice that as y_0 goes to zero, the x-integral and z-integral will both be identically zero, whereas the y-integral will remain a simple constant integral. This would be given by

$$B_y = \frac{\mu_0 I}{2R}$$

where R is the radius of the circle. That y-integral just gave a factor of 2π for the whole circle, whereas any fraction would likewise give a fraction of 2π. So for contributions to the B-field at the center of a circle, it is only a matter of (1) finding the fraction, and multiplying the above equation by that fraction, and (2) deciding whether the contribution is positive or negative by using the right hand rule.

Right hand rule for currents If the right thumb points in the direction of the current, the right fingers waggle in the direction of the generated magnetic field.

From this, it can be seen that the radius 1 semi-circle makes a negative contribution, while the radius 2 quarter-circle makes a positive contribution. (Positive-z is up out of the paper) The answer, therefore, is

$$B = \frac{-1}{2}\frac{\mu_0 I}{2\cdot 1} + \frac{1}{4}\frac{\mu_0 I}{2\cdot 2} = -\frac{3}{16}\mu_0 I$$

Chapter 6

Magnetic Induction

Changing magnetic fields can create electric fields. These electric fields, in turn, drive currents. This principle is what lies behind power generation in the modern world.

6.1 Faraday's Law

It turns out that the magnetic fields don't even have to change with time: all that is required is that the magnetic flux through a certain region changes. The magnetic flux, ϕ_M, is defined by

$$\phi_M = \int \vec{B} \cdot d\vec{A} \tag{6.1}$$

The unit of magnetic flux is the **Weber**. Notice that the definition is identical to that of electric flux.

Changing flux through a region creates an electric field, which in turn creates an electric potential. Quantitatively, this is given by Faraday's Law:

$$EMF = -\frac{d\phi_M}{dt} \tag{6.2}$$

If there are N loops, then each loop will get a boost equal to that calculated in the above equation, just like having N batteries in series. The electric field can be calculated from

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\phi_M}{dt} \tag{6.3}$$

In these calculations, orientation of the surface is important.
Rule of Thumb: If the right thumb is pointing in the direction of the normal to the surface, $\vec{A}$, then the direction of positive current flow is in the direction of the fingers.
Lenz's Law In a situation involving a changing magnetic flux, the induced current moves in such a way so as to create a counter-B-field, which will oppose any change.

6.2 Motional EMF

Motion of through a magnetic field can generate potential differences. In some books there is an attempt to relate this to some kind of changing flux through an infinitesimal region, but in fact it is really a result of the Lorentz Force Law,

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

When metal is moved through a magnetic field, the conduction electrons will respond to the magnetic field, creating an electrostatic field, continuing until the resulting electric force balances exactly the magnetic force. Hence, from the Lorentz force,

$$\left(\vec{E} + \vec{v} \times \vec{B}\right) = 0 \rightarrow -\vec{E} = \vec{v} \times \vec{B}$$

Integrating both sides:

$$-\int \vec{E} \cdot d\vec{s} = \int \vec{v} \times \vec{B} \cdot d\vec{s}$$

At this point, notice that the left hand side of the above equation is by definition the potential difference between two points. Thus:

$$\Delta V = \int \vec{v} \times \vec{B} \cdot d\vec{s} \tag{6.4}$$

6.3 Self-Inductance

6.3.1 Inductors

Inductors are coils of wire, as in solenoids, or any other geometry that creates a signficant magnetic field as current goes through it. Note that since magnetic fields are always created by flowing current, every circuit has some inductance effects. The voltage drop across an inductor is given by

$$\Delta V = -L\frac{dI}{dt} \tag{6.5}$$

L is called the inductance, and is defined by

$$L = \frac{N\phi_B}{I} \tag{6.6}$$

where N is the number of loops in the inductor, I is the current, and ϕ_B the magnetic flux. The SI unit of inductance is the **Henry**. Inductors are also sometimes called 'chokes' because they choke off current. The developing magnetic flux, by Lenz's law, creates an induced counter-magnetic field that opposed the change in flux. This can be easily seen in the next example.

6.3.2 Energy in an Inductor

The energy in an inductor is easy to calculate.

$$\Delta V_{ind} = L\frac{di}{dt} \rightarrow P = i\Delta V = L\frac{idi}{dt}$$

where P is the power. Note that $P = dU_L/dt$, so

$$U_L = \int Lidi = \frac{1}{2}Li^2 \tag{6.7}$$

6.4 RL, LC, and RLC circuits

6.4.1 RL Circuits

Suppose an inductor, resistor, and battery are in series in a closed circuit, with an open switch. At t=0 the switch is closed, so current can begin to flow. Using Kirchoff's laws and elementary differential equations, we can find the behavior of the circuit at any time. First, apply Kirchoff's law to the circuit. This results in

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

This can be massaged into an easily separable form, as follows. First, get dI/dt on one side, then factor out a $-R/L$.

$$\frac{dI}{dt} = \frac{1}{L}(\mathcal{E} - IR) = -\frac{R}{L}\left(I - \frac{\mathcal{E}}{R}\right)$$

Dividing through by $I - \mathcal{E}/R$ and 'multiplying' by dt gives the separated equation:

$$\frac{dI}{I - \mathcal{E}/R} = -\frac{R}{L}dt$$

This first-order equation can be easily integrated:

$$\ln(I - \mathcal{E}/R) = -\frac{R}{L}t + const \rightarrow I = \frac{\mathcal{E}}{R} + be^{-t/\tau}$$

where b is a constant depending on initial conditions and $\tau = L/R$ is called the time constant. At t=0 we have

$$I(0) = 0 = \frac{\mathcal{E}}{R} + b \rightarrow b = -\frac{\mathcal{E}}{R}$$

So finally,

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)$$

From this equation it can be seen that as time progresses, the choke relaxes with characteristic time τ, until after a very long time the current corresponds to that produced by a lone resistor in the circuit. This is because, eventually, a steady current is reached, at which time the voltage drop across the inductor is zero.

It is also possible to solve the case where this same circuit has the battery suddenly cut out of the loop. At first glance, it appears nothing much would happen, but in fact the magnetic field in the solenoid would start to drop, and this would create a changing voltage drop across the inductor. Using the same analysis as before will result in an exponentially decaying current:

$$I(t) = I_0 e^{-t/\tau} \tag{6.8}$$

where again $\tau = L/R$.

6.4.2 LC Circuits

Capacitors and inductors, of course, can be put in the same circuit together. An interesting oscillator consists of a fully charged capacitor put in a loop across an inductor. The capacitor then alternately discharges and charges, while the magnetic field alternately grows and decays. Kirchoff's law gives

$$-\frac{Q}{C} - \frac{dI}{dt} = 0 \rightarrow \frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

This is, of course, the harmonic oscillator equation. The solution is

$$Q = Q_0 \cos(\omega t + \delta)$$

where $\omega^2 = 1/LC$. In the event where at t=0 the capacitor has its maximum charge, it is evident that $\delta = 0$ and that the charge on the capacitor goes through harmonic oscillations forever. Naturally, this is an idealization, and there are always resistive effects that will bleed off the energy. In addition, such electromagnetic oscillations result in the production of electromagnetic waves—light.

6.4.3 RLC circuits

The RLC circuit is more realistic than the LC circuit, because in practice there is almost always some resistance in a circuit. For a battery, inductor, capacitor, and resistor in series, Kirchoff's law gives

$$\mathcal{E} - L\frac{dI}{dt} - \frac{Q}{C} - IR = 0$$

Using $I = dQ/dt$ and substituting, arrive at the following differential equation (after a little algebra)

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}$$

This equation is easy to solve, being an inhomogeneous ordinary differential equation. The solution is the sum of a particular solution, Q_p, and a general solution to the corresponding homogeneous equation. Homogeneous just means the right hand side is zero, rather than some constant or function of time. For the particular solution, we guess, with the only criteria being that when we plug the guess into the left hand side, we get the right hand side (in this case the constant $\mathcal{E}$. Obviously,

$$Q_p = \mathcal{E}C$$

Plugging this in, indeed, satisfies the equation. Now we need the general homogeneous solution. This can be obtained by the standard trick of finding the roots of the "characteristic equation". Essentially, this is simply a matter of plugging $Q = e^{mt}$ into the equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

and then canceling out the exponentials after taking the derivatives. Recall that the derivative of the exponential is always the exponential back again, times any chain rule factor. We have

$$Lm^2 + Rm + \frac{1}{C} = 0$$

This can be solved for m using the quadratic formula:

$$m = \frac{-R \pm \sqrt{R^2 - 4 \cdot L \cdot (1/C)}}{2L} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

This equation has three possible results, depending on whether the quantity under the radical is zero, positive, or negative.

Case 1: $\frac{R^2}{4L^2} - \frac{1}{LC} = 0$ This is called the critically damped case. Here, there is a double root, m=-R/2L, so the homogeneous solution is

$$Q_h = \alpha e^{-Rt/2L}$$

The complete solution is

$$Q = Q_p + Q_h = \mathcal{E}C + \alpha e^{-Rt/2L}$$

α, of course, is a constant, determined by the initial conditions.

Case 2: $\frac{R^2}{4L^2} - \frac{1}{LC} > 0$
In this case, there will be two distinct solutions, both of the exponentials having negative powers. The solution is

$$Q = Q_p + Q_h = \mathcal{E} + e^{-Rt/2L}\left(\alpha e^{\gamma t} + \beta e^{-\gamma t}\right)$$

where

$$\gamma = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Case 3: Underdamped $\frac{R^2}{4L^2} - \frac{1}{LC} < 0$
In this case, we have a negative under the radical, which means the solutions involve imaginary numbers. The solution is therefore

$$Q = Q_p + Q_h = \mathcal{E} + e^{-Rt/2L}\left(\alpha e^{i\omega t} + \beta e^{-i\omega t}\right)$$

where

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Now, it turns out that it can be shown–with power series expansions–that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Using this fact, it is possible to rewrite the expressions with imaginary numbers in terms of real numbers. Notice that we can make the constants α and β anything we want in order to eliminate all traces of imaginary numbers, which are considered unphysical. Obtain:

$$Q = Q_p + Q_h = \mathcal{E} + e^{-Rt/2L}\left(a\cos\omega t + b\sin\omega t\right)$$

Here, it can be seen that ω is the angular frequency of the oscillation, modified from what it was in the LC circuit by the resistance, R. Often, ω will be written as

$$\omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}}$$

to emphasize its relationship to the LC circuit.

6.5 Mutual Inductance

Mutual inductance refers to pairs of loops that affect each other through inductance. The idea is really the same as in self-inductance, but a little confusing, because there are always two parties involved, and each has a constant associated with it similar to the constant L, usually denoted M, with subscripts. However, it can be proven that for the two loops, loop 1 and loop 2, the mutual inductance of loop 1 is the same as the mutual inductance of loop 2, which is usually expressed as $M_{12} = M21$. Mutual inductance is important in the theory of transformers, which allow transmission of power at very high voltage but low frequency, followed by conversion to smaller voltages at the far end of the line for household applications. This saves in energy losses, since transmission at high voltage is much more efficient. We have

$$\mathcal{E}_\infty = -M\frac{dI_2}{dt} \tag{6.9}$$

and

$$\mathcal{E}_\in = -M\frac{dI_1}{dt} \tag{6.10}$$

The induced EMF in loop 1, say, is due to the contributions from loop 2 and self-contributions, from its own current.

$$\mathcal{E}_\infty = \mathcal{E}_{11} + \mathcal{E}_1 2 = -L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt} \tag{6.11}$$

6.6 Examples

Example 1: Mega-Example
Suppose a circle of wire in the x-y plane of radius 2 m contains a changing magnetic field given by $\vec{B} = 3e^{-2t}\hat{z}$. (A) Find the induced EMF in the coil after two seconds. (B) Find the electric field for $r < 2$. (C) Find the electric field for $r > 2$. (D) If the resistance of the wire loop is 5 ohms, find the current. (E) Find the direction of the current. (F) If the single loop of wire is replaced by five tight loops of the same radius, find the EMF. (G) In this instance, find the current.

Solution: These are all straightforward applications of the Faraday and Lenz laws, along with Ohm's law.
(A) Faraday's Law does the trick. First, calculate the magnetic flux:

$$\phi_M = \int \vec{B} \cdot d\vec{A} = \vec{B} \cdot \int d\vec{A} = 3e^{-2t}\hat{z} \cdot (\pi 2^2)\hat{z} = 12\pi e^{-2t}$$

$$\mathcal{E} = -\frac{d\phi_M}{dt} = 24\pi e^{-2t}$$

(B) Here, just as with Gauss's Law and Ampere's Law, only the flux within a certain radius counts–everything outside of that doesn't contribute.

$$\oint \vec{E} \cdot d\vec{s} = E(2\pi r) = -\frac{d\phi_M}{dt} = 6\pi r^2 e^{-2t} \rightarrow E = 3re^{-2t}$$

(C) Outside:

$$E(2\pi r) = 3\pi 2^2 e^{-2t} \rightarrow E = 6\frac{e^{-2t}}{r}$$

(D)

$$I = \mathcal{E}/R$$

(E) The direction of the current can be found in two ways: either by Lenz's law or by careful arithmetic. Lenz's law says that the induced current creates a magnetic field that attempts to counteract the change in magnetic flux. Therefore the current flows in the standard counterclockwise direction (x-axis to y-axis and on around) since this, by the right hand rule, will create a B-field going up through the loop. This up-going field will fight the decrease in flux.

Careful math will also give the same answer. By the rule-of-right-thumb, stick the thumb in the direction of the area vector, $\vec{A}$. The fingers then give the direction of positive current for the loop. Mathematically, part A gave a positive answer for $\mathcal{E}$, hence division by R gives a positive answer–which means

the current flows in the positive direction, as determined by the right-hand-rule. This is the same answer as before.
(F) N loops will give N times the EMF. However, if too many loops are added, the induced magnetic field will start to perturb the applied external field, reducing it significantly, and choking off further EMF enhancement.
(G) The current isn't changed, since the resistance also increases by a factor of N.

Example 2: Airplanes and motional EMF
Suppose an airplane with wingspan of 10 meters, tip-to-tip, is flying east at 300 m/s. If the Earth's magnetic field is given at this latitude by $(0, 2, -3) \times 10^{-4} T.$, what is the induced voltage across the wings?

Solution: The airplane is traveling east, which will be taken as the x-direction. North is then the y-direction, and straight up the z-direction. The wings are oriented south-north, which is in the y-direction. The solution, therefore, may be had by first computing the cross product, then integrating across the wings.

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 300 & 0 & 0 \\ 0 & 2 & -3 \end{vmatrix} \times 10^{-4} = (0, 0.09, 0.06)$$

Also, it is clear that $d\vec{s} = (0, dy, 0)$. (This comes from the parametrization (0,y,0).)

$$\Delta V = \int_0^1 0(-, 0.09, 0.06) \cdot (0, dy, 0) = \int_0^1 00.09 dy = 0.9 volts$$

Example 3: Faraday's Disk
A metal disk of radius R is spun on its axis at ω radians per second. A magnetic field $\vec{B}$ is oriented perpendicular to the disk. Find the induced ΔV between the center and the edge of the disk.

Solution: Orient the axes so that the z-axis goes through the center of the disk, and the disk is in the x-y plane. Focus on a typical element of the disk on the y-axis. Then on the y-axis,

$$\vec{v} = (0, -y\omega, 0)$$

and

$$\vec{B} = (0, 0, B)$$

The cross product is easy:

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -y\omega & 0 & 0 \\ 0 & 0 & B \end{vmatrix} = (0, \omega B y, 0)$$

Integrate this from y=0 to y=R:

$$\int \vec{v} \times \vec{B} \cdot d\vec{s} = \int_0^R \omega B y dy = \frac{1}{2} \omega B y^2 \Big|_0^R = \frac{1}{2} \omega B R^2$$

Example 5: An RL circuit
A 5 henry inductor is in a circuit with a 20 ohm resistor and 10 volt battery. (A) If the switch is closed at t=0 s., how long does it take for the current to reach half its maximum value? (B) what is the voltage drop across the inductor at that time?

Solution: This is a matter of plugging into equation. The maximum current will occur in the limit as $t \rightarrow infnty$, and be equal to $\mathcal{E}/R$ Let $I = 0.5\mathcal{E}/R$ and solve for time:

$$\frac{1}{2}\frac{\mathcal{E}}{R} = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)$$

Note that $\tau = L/R = 5/20 = 0.25\,s$. With a little algebra,

$$e^{-t/0.25} = \frac{1}{2} \rightarrow -4t = \ln(0.5) \rightarrow t = 0.173\,s$$

To solve the second part, either we have to find the derivative of the expression for the current and put that into the definition of the voltage drop across the inductor, or we can solve the easier problem of finding the drop across the resistor, then using Kirchoff's circuit law to get the drop across the inductor. We choose this latter method. First, find the current at that time:

$$I = \frac{1}{2}\mathcal{E}/R = 0.25\,V.$$

Next, the voltage drop across the resistor at this time:

$$\Delta V_{resistor} = IR = 0.25 \times 20 = 5\,V.$$

Finally,Kirchoff's law gives

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \rightarrow L\frac{dI}{dt} = \mathcal{E} - IR = 10 - 5 = 5\,V.$$

Of course, the answer makes perfect sense.

Example 5: Inductance of a Solenoid Find the inductance L of a solenoid with n wraps per unit length.

Solution: This is just a matter of assembling the pieces of the definition. From Ampere's law, the magnetic flux through the center of a solenoid is given by

$$\phi_B = BA = \mu_0 n I A$$

where A is the cross-sectional area. Inserting this into the definition and canceling the current gives the answer:

$$L = \frac{N\phi_B}{I} = N\mu_0 n A = \frac{N^2\mu_0 A}{\mathcal{L}}$$

Chapter 7

Electromagnetic Waves

7.1 The Wave Nature of Light

Light, in the large, can is created by accelerating charges, and can be imagined as something like taking a bed sheet and jerking it up and down, which causes ripples to propagate away as if independent of the sheet. The wave solutions come from Maxwell's Equations, which are the full expression of electromagnetism, together with the Lorentz Force Law containing all the information necessary to the solution of all classical electromagnetism problems.

7.2 Maxwell's Equations

We are now in a position to write down all of Maxwell's equations for electromagnetic fields in free space. These are:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{inside} \tag{7.1}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{7.2}$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\phi_M}{dt} \tag{7.3}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{inside} + \epsilon_0 \mu_0 \frac{d\phi_E}{dt} \tag{7.4}$$

Three things to notice:
1. Ampere's law now has an electric flux term. The previous version was only good in the absence of changing magnetic fields, when the currents were steady. The new form makes it more similar to Faraday's law.
2. A new magnetic law, given by the second equation in the group, similar to Gauss's law, but with zero on the right hand side. This difference is a

consequence of the fact that there are apparently no isolated magnetic charges– no north pole without a south pole. Unlike the electric field, which can have either positive or negative charges, there are no 'monopoles' of charge.
3. Other than remark 2, the two fields can be very nearly switched with little effect. Magnetic fields are similar to electric fields.

Finally, there is the Lorentz Force Law,

$$F = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{7.5}$$

These five equations are at the heart of electrodynamics.

7.3 Electromagnetic Waves

It can be shown fairly easily, by converting the integral equations into differential equations, that B and E satisfy wave equations. This suggests that light has wave nature, which in fact is well known. They can be shown to satisfy

$$\vec{E} = E_{max} \sin(\omega t - kx)\hat{u} \tag{7.6}$$

$$\vec{B} = B_{max} \sin(\omega t - kx)\hat{v} \tag{7.7}$$

where $\hat{u}$ and $\hat{v}$ are unit vectors that are perpendicular to each other. In the course of deriving these properties, two interesting discoveries can be made. First,

$$E = Bc \tag{7.8}$$

and second

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \tag{7.9}$$

where c is the speed of light.
The **Poynting Vector** is given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{7.10}$$

This vector always points in the direction of propagation of the electromagnetic wave, while its magnitude is equal to the energy flux of the wave.

7.4 Examples

Example 1: Magnetic field created by a discharging capacitor Find the induced magnetic field formed between the plates of a discharging parallel-plate capacitor with circular plates. Ignore edge effects.

Solution There is no current between the plates, so $I_{inside} = 0$. The charge on a discharging capacitor is given by

$$Q = Q_0 e^{-t/RC}$$

The electric field between the two plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0} = \frac{Q_0 e^{-t/RC}}{\epsilon_0 \pi a^2}$$

where a is the radius of the capacitor. The electric flux is just the total area contained inside the line integral, πr^2, where r is the radius at that point, times E. Ampere's law then gives

$$\oint \vec{B} \cdot d\vec{s} = B(2\pi r) = \epsilon_0 \mu_0 \frac{d\phi_E}{dt} = -\frac{\mu_0 Q_0}{RC} e^{-t/RC} \left(\frac{\pi r^2}{\pi a^2} \right)$$

Now, imagine a plane parallel to the two capacitor plates and going straight through the middle. At a point on this plane where $r > a$, the magnetic field would be

$$B = \frac{1}{2\pi r} \epsilon_0 \mu_0 \frac{d\phi_E}{dt} = -\frac{1}{2\pi r} \frac{\mu_0 Q_0}{RC} e^{-t/RC} \left(\frac{\pi a^2}{\pi a^2} \right) = -\frac{1}{2\pi r} \frac{\mu_0 Q_0}{RC} e^{-t/RC}$$

whereas for $0 < r < a$ the calculation yields

$$B = \frac{1}{2\pi r} \epsilon_0 \mu_0 \frac{d\phi_E}{dt} = -\frac{1}{2\pi r} \frac{\mu_0 Q_0}{RC} e^{-t/RC} \left(\frac{\pi r^2}{\pi a^2} \right) = -\frac{\mu_0 Q_0}{RC} e^{-t/RC} \left(\frac{r}{2\pi a^2} \right)$$

Example 2: A Plane Electromagnetic Wave
Find the equations for the magnetic and electric fields of a co-sinusoidal electromagnetic wave that is propagating in the x-direction, and which has, when x=0, t=0 takes its maximum electric field amplitude of 200 N/C in the positive y-direction, and an angular wave number of 10π *radians/meter*.

Solution: This is a matter of assembling several facts. First get angular frequency from the angular wave number.

$$k = 10\pi = 2\pi/\lambda \rightarrow \lambda = 1/5$$

$$c = f\lambda \rightarrow f = c/\lambda \rightarrow \omega = 2\pi c/\lambda = 3\pi \times 10^9 \; Hz$$

Next, get the amplitude of the B field with

$$E = Bc \rightarrow B = E/c = 6.67 \times 10^{-7} \; Tesla$$

Finally, the directions of the fields are needed. The electric field $\vec{E}$ is already determined because we are told it is initially in the $\hat{y}$ direction. The $\vec{B}$ field must be in a direction such that the Poynting vector, $\vec{E} \times \vec{B}/\mu_0$, points in the positive x-direction, the direction of propagation. This can be done algebraically with three components of an unknown unit vector representing the direction of the B-field, however it's easier using the right-hand-rule. Point the fingers in the y-direction of the electric field and curl in such a direction so that the thumb points in the positive x-direction, the direction of propagation. At t=0, x=0, it is found that the B-field must point in the positive z-direction. The E-field and B-fields can now be written down.

$$\vec{E} = E_max \cos(\omega t - kx)\,\hat{u} = 220 \cos\left(3\pi \times 10^9 t - 10\pi x\right)\hat{y}$$

$$\vec{B} = B_max \cos(\omega t - kx)\,\hat{v} = 6.67 \times 10^{-7} \cos\left(3\pi \times 10^9 t - 10\pi x\right)\hat{z}$$

Example 3: Direction of Propagation
Suppose a light wave has electric field given by $\vec{E} = (3, 4, 0)$ and magnetic field given by $\vec{B} = (0, 0, 5/c)$ at a certain instant of time. (A) Find the direction of propagation of the wave (B) Find the intensity of the wave.

Example 4: Energy Densities
Calculate the average electric and magnetic field densities 5 meters from a monochromatic light source emitting 300 watts of power.

Solution: This problem is complex and considered difficult only because there are so many definitions involved, and all of them have very similar terms. The electric energy density is given by

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

When inserting the typical co-sinusoidal expression for $\vec{E}$, a square in cosine results, which on averaging gives a factor of 1/2. So

$$u_{E_ave} = \frac{1}{4}\epsilon_0 E_{max}^2$$

Example 5: Solar Sails
A solar sail is in free space, at rest relative the sun, at the radius of the Earth's orbit. (A) If the area of the sail is one thousand square meters, and the total mass, including payload, is 10,000 kg, find the acceleration of the sail, assuming it's going radially outward. (B) Find the terminal velocity (the velocity at infinity). Note: the solar intensity is about $1,400\ watts/m^2 attheradiusofEarth'sorbit.$

Solution: There are a variety of ways to derive the pressure of light on a surface. The easiest is probably to use the result from relativity that E=pc, where p is the momentum. Rearranging and taking the derivative with respect to time yields

$$F = \frac{dp}{dt} = \frac{dE/dt}{c} = \frac{P}{c} = \frac{IA_s}{c}$$

where P is the power, I is the intensity, and A_s is the area of the sail . This corresponds to the case where the light is absorbed: if it is perfectly reflecting, there is double the kick: one kick when the photon hits, and a second when it pushes off and goes back the way it came. So in the perfectly reflecting case, the above equation gets a factor of 2. Actual solar sails will have varying multiplicative factors between 1 and 2. Now, since the sail will be changing position, it is necessary to have an expression for the intensity as a function of r, the radial distance from the center of the sun. This isn't hard to come up with, if we recognize that the total energy per second crossing any sphere centered on the sun is always the same–just spread out over a larger area. This means that

$$P = IA = I\left(4\pi r^2\right) = I_0\left(4\pi r_0^2\right) \rightarrow I = I_0\frac{r_0^2}{r^2}$$

The last expression is the one we need–notice that the intensity falls off like $1/r^2$ just like the force of gravity. I_0 is just the intensity at a given location, r_0, which in this case is the radius of the Earth's orbit around the sun. Newton's second law can now be written down:

$$ma = \frac{I_0 A_s}{c}\frac{r_0^2}{r^2} - \frac{mMG}{r^2}$$

This all boils down to

$$\frac{d^2r}{dt^2} = \left(\frac{I_0 A_s r_0^2}{mc} - MG\right)\frac{1}{r^2}$$

Example 6: Polarization
Unpolarized light traveling parallel to the y-axis is incident on a filter with optical axis in the z- direction. Transmitted light continues on, encountering another filter with axis tilted thirty degrees with respect to the positive z-axis, and then a final filter which is tilted 90 degrees with respect to the z-axis. What is the intensity of the light after passing through the last filter?

Solution: The first filter cuts the intensity in half. The second filter will tack a $\cos^2 30$ onto this, and the final filter will contribute a factor of $\cos^2 60$. Notice that the angle needed is the one with respect to the previous polarizer.

$$I = \frac{I_0}{2}\cos^2 30\cos^2 60 = \frac{3}{32}I_0$$

Chapter 8

AC circuits

Alternating current changes direction constantly, due to the method of generation, which comes from spinning loops of wire in fixed magnetic fields, for example. If ω is the angular frequency, then

$$i = i_0 \sin \omega t \tag{8.1}$$

The amplitude i_0 is called the **peak value** of the current. Other familiar quantities, such as the potential difference across some element of the circuit, also have peak values, and in general these don't occur at the same time, and are said to be out of phase.

A resistor in an AC circuit behaves in the obvious way. The instantaneous potential difference across the resistor is given by

$$v_R = i_0 R \sin(\omega t) \tag{8.2}$$

The peak value is, of course, $v_{0R} = i_0 R$. The power dissipated by the resistor is

$$p = i^2 R = i_0^2 R \sin^2(\omega t) \tag{8.3}$$

This expression can be averaged with the mean value theorem for integrals. Since the functional dependence is in the current, i, this is equivalent to averaging i^2, which results in

$$I = \sqrt{i_{ave}^2} = \frac{i_0}{\sqrt{2}} \tag{8.4}$$

I is called the **root mean square (rms) current**. Upper case letters, in general, will be used for *rms* values. It can then be easily shown that the *rms* values of V and P are given by

$$V = \frac{v_0}{\sqrt{2}} \tag{8.5}$$

$$V_R = IR \tag{8.6}$$

$$P = I^2 R \tag{8.7}$$

With the current still given by a sinusoid, it is possible to write down an equation describing the response of an inductor.

$$v_L = L\frac{di}{dt} = v_{0L}\cos(\omega t) = v_{0L}\sin\left(\omega t + \frac{\pi}{2}\right) \tag{8.8}$$

This means the potential drop across the inductor leads the current by $\pi/2$. A similar situation occurs with a capacitor in an AC circuit.

$$q = \int i_0 \sin(\omega t) = -\frac{i_0}{\omega}\cos(\omega t) = -\frac{i_0}{\omega}\sin\left(\omega t - \frac{\pi}{2}\right)$$

Using the definition of capacitance, this equation becomes

$$v_{0C} = -\frac{i_0}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right) \tag{8.9}$$

which shows that the potential drop across a capacitor lags the current by $\pi/2$.

Example 1: A series circuit Suppose an AC circuit has a 20 millihenry inductor in series with a 60 microfarad capacitor and a 30 ohm resistor. If the frequency is 60 Hz and the peak voltage is 100 V., find the impedance of the circuit and the phase angle.

Solution: This is a straight-forward calculation. First, calculate the angular frequency, then the inductive reactance and capacitive reactance.

$$\omega = 2\pi f = 200\pi = 628.3\ rad/sec$$

$$X_L = \omega L = 628.3 \times 20 \times 10^{-3} = 12.57\ ohms$$

$$X_C = 1/\omega C = 26.53\ ohms$$

The impedance is given by

$$Z = \left(R^2 + (X_L - X_C)^2\right)^{1/2} = \left(30^2 + (12.57 - 26.53)^2\right)^{1/2} = 33.09\ ohms$$

The phase angle is given by

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{12.57 - 26.53}{30} = -0.4563 \to \phi = 24.5\ deg$$

A condition called **RLC resonance** occurs when $X_L = X_C$. Then

$$X_L = \omega L = X_C = \frac{1}{\omega C}$$

This equality will occur for a particular value of the angular frequency, ω_p which by inspection can be seen to be $\omega_p = 1/\sqrt{LC}$. The current is then in phase with the voltage, and takes its maximum value, which is $V = I/R$. **Transformers** A transformer transforms power from one frequency to another, using magnetic induction. It consists of a primary and secondary coil each wrapped around the same iron coil, one inside the other. The primary coil is connected to an AC source. If the voltages are V_1 and V_2, and the corresponding wraps are N_1 and N_2, then the relationship between wraps and voltages is given by

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \tag{8.10}$$

Power transfer is nearly perfectly efficient, so that $P_1 = P_2$. This implies that

$$I_1 V_1 = I_2 V_2 \rightarrow \frac{I_1}{I_2} = \frac{V_2}{V_1} \tag{8.11}$$

which gives a relationship between the voltages and the currents.
Example 2: An ideal transformer has 300 turns in the primary coil and 100 in the secondary coil. Suppose that the primary coil has an rms current of 1.5 A. when the rms potential difference is 60 V. Find the rms potential difference and rms current in the secondary coil.

Solution: This is straightforward. Using the transformer equation, the voltage in the secondary can be calculated:

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} = \frac{100}{300} \rightarrow V_2 = \frac{1}{3}60 = 20\,V.$$

The current can be calculated from the relationship between currents and voltages.

$$\frac{I_1}{I_2} = \frac{V_2}{V_1} = \frac{20}{60} = \frac{1}{3} \rightarrow I_2 = 3I_1 = 4.5A.$$

Example 3: RLC Resonance

Suppose an RLC circuit has a resonance frequency of 500 Hz. If $X_L = 50\,ohms$ and $X_C = 10\,ohms$ at a certain frequency ω, find L and C.

Solution:

$$\frac{1}{\omega C} = X_C = 10 \quad \omega L = X_L = 50 \rightarrow \frac{L}{C} = 500$$

Meanwhile, the resonance frequency is such that

$$1000^2 = \frac{1}{LC} \rightarrow 500 \cdot 1000^2 = \frac{1}{C^2} \rightarrow C = 4.47 \times 10^{-5}\ Farads$$

Putting that back into the first equation gives $L = 0.0224\ H$.