The axiomatic skeleton for fuzzy set intersections $i(a,b)$ and unions $u(a,b)$ is given by:

1. boundary conditions: $i(1,1) = 1$; $i(0,1) = i(1,0) = i(0,0) = 0$; $u(1,1) = u(0,1) = u(1,0) = 1$; $u(0,0) = 0$
2. commutativity: $i(a,b) = i(b,a)$; $u(a,b) = u(b,a)$
3. monotonicity: if $a \le a'$ and $b \le b'$ then $i(a,b) \le i(a',b')$ and $u(a,b) \le u(a',b')$
4. associativity: $i(i(a,b),c) = i(a,i(b,c))$; $u(u(a,b),c) = u(a,u(b,c))$

In general, fuzzy set unions and intersections are not idempotent. Every possible selection of fuzzy union, intersection and complement violates some properties of the Boolean lattice. In particular, the law of excluded middle and the law of contradiction may fail to be satisfied.

It is known that the conditions 1–4 lead to the inequalities $i(a,b) \le \min(a,b)$ and $\max(a,b) \le u(a,b)$, so that $i(a,b) < u(a,b)$ if $a \neq b$. The most common form of the fuzzy set complement is $c(a) = 1 - a$.

The following example shows that the axiomatic skeleton 1–4 may be incompatible with DeMorgan's laws:

$$u(a,b) = c(i(c(a),c(b))); \quad i(a,b) = c(u(c(a),c(b)));$$

Let us assign to every element x_i of a fuzzy set $A = \sum_i \mu_i / x_i$ a number $\chi_i \in (-\infty,+\infty)$ such that $\mu_i = \frac{1}{2}\left(1 + \mathrm{th}(\chi_i)\right)$.

With χ varying from $-\infty$ to $+\infty$, μ monotonically increases from 0 to 1. Let the complement of A be constructed according to the usual rule:

$$c(\mu(\chi)) = 1 - \mu(\chi) = \tfrac{1}{2}\left(1 - \mathrm{th}(\chi)\right) = \tfrac{1}{2}\left(1 + \mathrm{th}(-\chi)\right) = \mu(-\chi)\,.$$

Let us define the intersection of two fuzzy sets with

$$i(\mu_1(\chi_1),\mu_2(\chi_2)) = \mu(\chi_1 + \chi_2) = \tfrac{1}{2}\left(1 + \mathrm{th}(\chi_1 + \chi_2)\right) = \mu_1\mu_2/(1 - \mu_1 - \mu_2 + 2\mu_1\mu_2)\,.$$

This expression satisfies the boundary conditions for the fuzzy set intersection and is evidently commutative, monotonic and associative. Associativity may also be checked directly, giving

$$i(i(\mu_1(\chi_1),\mu_2(\chi_2)),\mu_3(\chi_3)) = i(\mu_1(\chi_1),i(\mu_2(\chi_2),\mu_3(\chi_3)) = \frac{\mu_1\mu_2\mu_3}{(1 - \mu_1 - \mu_2 - \mu_3 + \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)}\,.$$

Thus defined $i(\mu_1,\mu_2)$ is however not idempotent, thought it is *asymptotically* idempotent at $\chi \to \pm\infty$, which ensures the correct transition to the ordinary (crisp) sets.

Now, let us use the DeMorgan's law expressing the union operation through the intersection and the complement:

$$u(\mu_1(\chi_1),\mu_2(\chi_2)) = c(i(c(\mu_1(\chi_1)),c(\mu_2(\chi_2)))) = c(i(\mu_1(-\chi_1),\mu_2(-\chi_2))) =$$
$$= c(\mu(-\chi_1 - \chi_2)) = \mu(\chi_1 + \chi_2) = i(\mu_1(\chi_1),\mu_2(\chi_2))$$

Since the union and intersection coincide, the union will not satisfy the axiomatic skeleton, violating the boundary conditions at $\mu_1 = 0$ and $\mu_2 = 1$. Of course, the value of $i(\mu_1,\mu_2)$ can never be less than $u(\mu_1,\mu_2)$ as it should be from the skeleton axioms. It should be noted, that the laws of excluded middle and contradiction are not satisfied for thus defined intersection/union: $i(\mu(\chi),\mu(-\chi)) = ½$.

The physical sense of the above definitions is relativistic addition of velocities. The particles moving forward with the speed of light are assigned with $\mu = 1$, while those moving with the speed of light in the opposite direction are assigned with $\mu = 0$. Note that the definition of the union reproducing the addition of positive velocities only

$$\mu(\chi) = \mathrm{th}(\chi_1 + \chi_2); \quad u(\mu_1(\chi_1),\mu_2(\chi_2)) = \mu(\chi_1 + \chi_2) = \mathrm{th}(\chi_1 + \chi_2) = (\mu_1 + \mu_2)/(1 + \mu_1\mu_2)$$

is known as a function of the Hamacher class with $\gamma = 2$, and, together with the standard complement, it produces the intersection $i(\mu_1,\mu_2) = \mu_1\mu_2/(1 + (1 - \mu_1)(1 - \mu_2))$, so that both the axiomatic skeleton and DeMorgan's laws are satisfied. However, the complement is "physically incompatible" with the union in this case: Galilean velocity addition gets mixed with the relativistic rule.