A few comments on
Cognitive (Semantic) Visualization of the Continuum Problem and Mirror Symmetric Proofs in the Transfinite Numbers Theory
by A.A.Zenkin
http://www.mi.sanu.ac.yu/vismath/zen/index.html
(version of October, 2000)

1. The paper could only be considered as a popular account on a few results from nonstandard analysis, without any new results obtained by the author, or new ways of obtaining earlier known results.

2. The Continuum Hypothesis (CH), in its classical formulation, is the assertion of impossibility of a set having cardinality between $\aleph_0$ and C [E.Engeler, *Metamathematik der Elementarmathematik* (Springer, 1983)]. Zenkin's formulation (1), $C = \aleph_1$, could be accepted as a formulation of CH only after a proper definition of $\aleph_1$, in the current context; in particular, it assumes that cardinalities form a discrete sequence, which may be not necessarily true. Similarly, Zenkin's formulation (2), that $\forall\alpha\, |P(\aleph_\alpha)| = \aleph_{\alpha+1}$, in some contexts may merely be a formulation of a theorem, rather than a problematic statement;
there is no reason why power set construction should be the only way of obtaining higher-cardinality sets.

3. The paper can hardly pretends to do with CH, since the existence of a 1-to-1 correspondence between a set of hyper-integers and the set of real numbers cannot be considered a re-formulation of the CH, as Zenkin suggests. This is an independent statement that may equally hold either with, or without CH.

4. The representations of real numbers from the segment [0, 1], as given by Zenkin, are mainly of an illustrative value, since they do not imply any properties specific for real numbers proper. Thus, geometrical points on the real axis may represent anything at all, being used as abstract labels; for instance, when used as hash values for randomly spread entities, they do not convey any continuity and order, characteristic of real numbers. In general, arithmetization of geometry is a non-trivial undertaking, and, in some spaces, or manifolds, it cannot be done with real numbers at all. The representation by an infinite sequence of binary digits is nothing but a conventional (vulgar) format for expressing the sum over the powers of 2, which has, for unknown reasons, been put as a separate representation in Zenkin's paper. In any case, constructing such sums already assumes the existence of real numbers, with all their properties, so that the definition of infinite summation could be introduced (as a limit). As for the binary tree path representation, it may certainly have a didactic value, provided it is combined with a reference to some real activities, involving binary discrimination. Obviously, we do not need to stick to the binary system, to construct real numbers, and each system has its own advantages and problems. There are also other ways of representing real numbers such as π, e, γ, $\sqrt{2}$, $\sin(\alpha n)$ etc.

It would be much more important to stress that any idea of a real number virtually implies a situation of choice, drawing a border between properly ordered entities. That is, a real number is an abstraction of (binary) discrimination, a mathematical model of the scheme of a real activity. The two main cases of choice are: (1) spatial (parallel) discrimination, ordering the simultaneously given entities, and (2) serial discrimination, deciding on the next action in a sequence. The former leads to the standard Dedekind definition of a real number as a boundary between rational numbers; the latter results in various serial representations, like a sequence of binary digits. However, a real number is always a continuing activity, rather than a finite action producing a definite product — as soon as we stop, we do not have a real number, but only its rational approximation [H.M.Hubey, *The Diagonal Infinity* (World Scientific, 1998)]. This is like life is nothing but living, and stopping it we'll get no more than a dead body.

5. Using the binary tree (or power sum) representation for "cognitive visualization" may be both productive and misleading. This is in the nature of any scheme at all, which cannot make any sense without a proper interpretation and specification. Thus, mere mirror reflection of the binary tree representing the real numbers in [0, 1], and constructing sums in positive powers of 2 rather than negative ones, is nothing but an informal operation, a kind of metaphor that still has to be justified, to become anything valuable. Even less justifiable is

the extension of the binary tree (power sequence) to the transfinite numbers, using the constructions like $2^{-\omega}$, $2^{-(\omega+1)}$, etc. — such construction do not make any sense until they are related to something formally correct; it should also be noted, that sequences of transfinite numbers can hardly be productive enough, since there is a continuum of transfinite numbers, as Zenkin himself indicates.

6. Employing binary (decimal, octal, ...) expansions for real numbers may be confusing, since there may be many sequences converging to the same limit. Thus, two sequences of partial sums

$$x_n = \sum_{k=1}^{n} a_k 2^{-k} \text{ and } x'_n = \sum_{k=1}^{n} b_k 2^{-k}$$

with $a_1 = 1$, $a_k = 0$ for $k > 1$, and $b_1 = 0$, $b_k = 1$ for $k > 1$, converge to the same real number 1/2, and there may be other infinite sums converging to the same limit. In the binary tree picture, sums S and S' correspond to quite different paths. In a system with a different base, different numbers may have such ambiguous representations. This indicates that real numbers are primary to any their representations, which, logically, can only be validated using the known properties of real numbers, and not in the opposite way, the properties of real numbers derived from their representations.

7. The fact that some finite sequences of binary digits can be re-interpreted as integers, using sums

$$\bar{x}_n = \sum_{k=1}^{n} a_k 2^{k}$$

instead of

$$x_n = \sum_{k=1}^{n} a_k 2^{-k} \ ,$$

does not mean that the same inversion operation should be applicable to any sequence at all. Nearly all mathematics is about specifying the conditions under which a particular statement holds, and finding an exact formulation of a theorem is often equivalent to proving it.

Formally, for any sequence $\{a_k\}$ of binary digits such that, for any integer n, there is an integer $k > n$ such that $a_k = 1$, the sequence of partial sums $\bar{x}_n$ does not converge to any limit, having no finite condensation points. I would call such sequences *infinite*, in contrast to the *finite* sequences, with $a_k = 0$ for all k greater than some integer n (the *length* of the sequence) — in Zenkin's terms, finite sequences have a 0-tail. Under certain conventions, infinity ∞ can be considered as the common limit of the sums $\bar{x}_n$ for all the infinite sequences — and we know that many sequences can converge to the same limit. In this (standard) interpretation, there is no need in transfinite integers, and no logical problems.

However, one can be interested in objects other than numbers, and study binary sequences, or paths in a binary tree, on themselves. Grouping such sequences into equivalence classes and defining the standard arithmetic operations on the collection of such classes may produce number-like constructs, transfinite numbers. However, doing it is not as trivial as Zenkin tries to demonstrate, requiring the notion of an ultra-filter, the cornerstone of nonstandard analysis [M.Davis, Applied Nonstandard Analysis (Wiley, 1977)].

8. Zenkin's Theorem 2 exploits notions and statements of nonstandard analysis in a circular way, and hence cannot be considered as rigorous enough. Comparison of divergent sequences is not well-defined in standard analysis, and playing on this indeterminacy may lead to any result at all. To use transfinite numbers, and compare them with other numbers, requires a good job of properly defining the possible operations, and the limits of their applicability. In the context of Zenkin's paper, one would rather use

$$\bar{x} = \lim_{n\to\infty} \sum_{k=1}^{n} a_k 2^{k}$$

to define the infinite ordinal ω, provided the limit is the same for any infinite sequence $\{a_k\}$.

9. Since $\bar{x}$ is not a set, while ordinal and cardinal numbers are defined only for sets, Zenkin's assertions that Ord $\bar{x} = \omega$ and Card $\bar{x} = \aleph_0$ do not look too convincing. The validity of identifying numbers with sets, as in [E.Mendelson, *Introduction to Mathematical Logic* (Van Nostrand, 1964)], is arguable, since it leads to logical paradoxes, and there may be different conceptual stands in that question. In general, comparing infinities can hardly be said to be a common convention, and it obviously depends on the acceptance of CH.

10. The definition of (in)equality of transfinite numbers via the (in)equality of the mirror real numbers lacks consistency. Thus, as indicated above sequences .1000... and .0111... converge to the same real number, while the mirror number of the former is finite and the mirror number of the latter is transfinite. There may be other, more intricate cases of that sort.

11. The definition of transfinite integers "with confinal heads" is not easily visualizable in the binary tree picture. It is much simpler to formally define a galaxy G_n as a set of all binary sequences that coincide for all $k > n$. Also, Zenkin does not give any definition of the difference of two transfinite numbers, and his statement that the difference of any two numbers in a galaxy is finite becomes arbitrary. In the binary expansion representation adopted by Zenkin, comparison and arithmetical operations can hardly be defined in a trivial way.

The structure of the class of transfinite numbers is, in fact, much richer than exposed in Zenkin's paper. Thus, the definition of a galaxy can be extended to account for general confinality of transfinite numbers:

> two sequences $\{a_k\}$ and $\{b_k\}$ of binary digits are called *confinal* iff there are integers m and n such that $a_{m+k} = b_{n+k}$ for any $k > 0$.

This means that, after a shift by $m–n$ positions, the sequences belong to the same galaxy — this is a kind of linear dependence, and the existence of a basis set is an interesting question.

It would also be interesting to study the properties of the *normalized* partial sums

$$\bar{r}_n = \frac{1}{2^n}\sum_{k=0}^{n-1} a_k 2^k = \sum_{k=0}^{n-1} a_k 2^{-(n-k)} = \sum_{k=1}^{n} a_{n-k} 2^{-k}$$

instead of $\bar{x}_n$. Obviously, $0 \le \bar{r}_n < 1$, but, in general, $\bar{r}_n$ does not converge to any real number r from [0, 1], since

$$\bar{r}_n - \bar{r}_{n+1} = \sum_{k=0}^{n-1} a_k 2^{k-n} - \frac{1}{2}\sum_{k=0}^{n} a_k 2^{k-n} = \frac{1}{2}\sum_{k=0}^{n} a_k 2^{k-n} - \frac{1}{2} a_{n+1} = \frac{1}{2}\left(\bar{r}_n - a_{n+1}\right)$$

so that the difference may become infinitesimal only in the case of $a_k = \text{const}$ for all k greater than some $K \ge 0$. As one could see, this transformation is an attempt to "invert" the binary sequence representing a real number from [0, 1], to count back from infinity.

12. The statements in Section 6 of Zenkin's paper are poorly formulated and most of them make no sense at all. Primarily, this concerns the presence of the $\omega+1$ level in the tree, which has no definition at all, and no justification. Accordingly, the phrases about a path "attaining" that level are nothing but empty words.

13. All the religious stuff from Leibniz quoted in the concluding section is irrelevant to the problems discussed, and to mathematics in general. Allusions to religion have been introduced in the paper to conform with the current political situation, rather than for any sensible reason.

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