A few comments on
Fatal Mistake of Georg Cantor
by A.A.Zenkin
http://www.com2com.ru/alexzen/WEB-2000/Cantor/VF_paper_English.html
(version of November, 2000)

1. There is a common term confusion about mathematics, logic, philosophy etc. Philosophy is thought of as a science, and hence is vainly expected to produce "theorems" and "truths"; logic (a part of philosophy) gets mixed with mathematics, and mathematical logic (a part of mathematics) is believed to be able to prove statements, and not only model the actual proofs... There is also a poorly defined idea of meta-mathematics, which is partly mathematics, partly philosophy—or whatever one might fancy. The blind belief in the ultimate truth of any proof lives together with the inability to distinguish proofs from any serial discourse at all.

Things are much simpler: science can construct abstract models of human activities, while the ways of constructing these abstractions and applying them back to real life are entirely determined by philosophy (ideology). Since science is a kind of activity, one can construct abstract models of science too, but it is only in the special branch of philosophy (methodology of science) that such models may acquire any applicability. In particular, mathematics is a science developing abstract models of the quantitative aspects of human activities—however, mathematics can never tell what quantity is, and which aspects should be considered as quantitative. Logic is a branch of philosophy that controls the application of the general schemes of activity (and formal reasoning in particular). Since logic and philosophy are also human activities, they may be modeled in science (e.g. in mathematics), producing yet another abstract scheme, which may (or may not) be applied to some activities within a certain logic.

2. The complexity of a proof is inversely proportional to its practical value. A proof 100 pages long can hardly ever be useful anywhere but in the field it has been originally made for. A proof 1000 pages long cannot be considered as a proof at all, since nobody will be able to properly check it and indicate all the assumptions made. The shorter is the proof, the more applicable its scheme may become to many more problems in numerous sciences, thus transforming into a logical scheme, a methodological (ideological) position.

As any science, mathematics is of no use on itself, and it must be applied (usually in science and technology), to make sense. There is no use discussing the "fundamental" problems that do not influence the human ways of life. One of my school teachers once said that the Great Fermat Theorem is of no use to anybody but the mentally ill. He may have been right.

3. The famous Cantor theorem about non-countability of real numbers is indeed one of the greatest achievements of the humanity, a concentrated expression of the conceptual problems hidden in the foundations of mathematics; it could be used to illustrate many existing metamathematical constructions, and it still provokes people to invent more.

From the viewpoint of standard mathematics, which simply makes use of its axioms and does not reflect on them any longer than needed to get convinced in their productivity and consistency, this is a rigorous theorem. That is, the uncountability of real numbers can be used in most cases as a well-established fact, without expecting any trouble. However, since no abstraction is universally applicable, there must be cases invalidating the principles on which the formulation and proof of the Cantor theorem are based. This does not mean that the theorem is not true—this only confines a general statement to its domain.

As it seems, the two possible directions of extending mathematics in a non-standard domain are:

> 1) Using a different logic — for instance, the proof of the falsity of the negation of a statement does not necessarily mean the truth of the statement itself: if a wolf has eaten your enemy, it does not mean that the wolf has become your friend; diagonal proofs become impossible in a logic like that.

2) Using a different idea of infinity — in Cantor's theorem, it is assumed that one can do something with *every* element of an infinite sequence, and obtain another infinite sequence *of the same class*; this means acceptance of the Choice Axiom, which is still disputable.

Certainly, there are other associated problems, such as the usage of a particular representation of an object for proving statements about the object in general, regardless of any representation. Also, one might question the principle of contradiction implying falsity. None of these and other real problems have been discussed in Zenkin's paper, which only treats a pseudo-problem caused by misunderstanding and lack of logic.

3. The principal fault of Zenkin's idea of applying the diagonal proof to finite sequences is that the construction of the diagonal sequence *of the same class* is impossible for finite sets. Considering finite sequences of length N, we obtain a set S_N containing 2^N elements; this means that constructing a diagonal sequence by any N elements will give yet another element of the same set, without any conceptual difficulties—however, one cannot construct a diagonal sequence of length N using *all* the 2^N elements of S_N , as in the case of the set of infinite sequences. This is a well-known characteristic of an infinite set, which makes it different from any finite set: there may be 1-to-1 correspondence between the set and its proper subset. In the informal language: if you add one to infinity, you still get the same infinity, while adding one to a finite number will produce a different finite number. In classical mathematics, infinity and zero are the two special objects that behave differently from any finite numbers; thus, if you multiply any non-zero number by two (or add it to itself), you get a different number, while multiplying zero by any finite number (multiply adding it to itself) will give nothing but zero. Similarly, the empty set and infinite sets are the two stumbling blocks for meta-mathematics.

That is, Cantor's proof essentially depends on the infinity of the set of natural numbers, and the infinity of the set of real numbers, contrary to Zenkin's assertion. As all classical mathematics, it certainly employs the actuality (existence) of the infinite sets, which does not contradict the practical observations, and the common schemes of activity. Of course, there may be situations, where no actual infinity could be possible, and a different mathematics would be needed.

4. Zenkin's attempts to construct a closure of the hypothetically enumerated set of real numbers adding the diagonal number to the enumeration would have been alright, had it not been assumed that the original enumeration already lists *all* sequences of binary digits. This is the contradiction that proves Cantor's theorem. Changing this assumption leads to a statement different from the original formulation of the theorem, so that any reference to Cantor becomes irrelevant.

In principle, the operational closure technique is quite acceptable in mathematics, and this how the space of real numbers can be constructed from rational numbers: just add all sequences that do not represent a rational number. Applying this technique to real numbers is also possible, but one should rather consider functions from $\Re$ onto $\{0, 1\}$, which would obviously produce the collection of all the subsets of the set of real numbers, with the cardinality of $\aleph_2$.

5. The logic of operational closure is incorrectly described by Zenkin as $\neg A \to B \to \neg B \to B \to \neg B \to \ldots$ To be more precise, one should write: $A \to \neg A \to B \to \neg B \to C \to \neg C \to \ldots$, with a new entity arising at each step. In dialectical logic, this is a normal sequence of any development, with B resembling A in certain respects, as the negation of negation; one could then say that rational numbers *develop* into real numbers via operational closure.

However, finite reasoning cannot grasp an infinite chain of negations like that, and hence Zenkin's conclusion that the process can never be concluded to produce an actually infinite set. This is why finite reasoning cannot be enough for a conscious being, who can fold any (potentially infinite) activity into a (finite) action. This occurs via the reflexivity of every activity: once it has been consciously comprehended, it becomes a separate entity, at a higher level of hierarchy, as compared to individual actions constituting it. In the case of rational numbers completed by irrational real numbers, the basic activity is enumeration—in a folded form, every enumeration corresponds to a real number. In practice, folding an activity implies finding some way of producing things in a finite time, rather than in an infinite approximation; thus, we can simply *draw* a continuous segment, moving a hand from one spatial point to another—then we can approximate (represent) this infinity with a finite number of pigment spots on the paper, or dots in the computer screen. A finite formula may well express an infinite motion, and the scheme of activity stay for the activity as an infinite process.

Transitions from finite entities to infinity occur quite often in real life and real reasoning, since any activity is essentially *hierarchical*. A conscious being can be simultaneously *doing something* (level of activity, non-finite)—and making something for that (level of action, finite). We can simultaneously observe a point in the continuous segment [0, 1] and construct discrete sequences of points converging to it. This is possible since the levels of activity are *qualitatively* different, and we distinguish them by their quality rather than quantity.

6. The enumeration of great names is no argument. One may (or may not) agree that Gauss, Kant, or Luzin, were as smart as Cantor, and knew as much (or as little) about infinity—however, this does not mean that they could not be wrong, or that Cantor's theorem could be annihilated by any other mathematical discovery. Also, Aristotle and Kant were philosophers, while Cauchy and Luzin were scientists, and Leibniz or Poincaré tried to combine science and philosophy in an eclectic way; they lived in different epochs and had very different attitude to the problems concerned, so that Zenkin's assertion that they all denied actual infinity seems very doubtful. Yet another Zenkin's assertion that science or technology never dealt with actual infinity is no less arguable. It is much more probable that science and technology (and any other activity) deal with infinity all the time, since every object is hierarchical, developing ever new levels, and hence presenting itself to the subject as both finite and infinite—the schemes of human reasoning merely reflect this circumstance in the ideas of finitude, potential or actual infinity.

7. Trying to reduce mathematics (or any other science) to the operational level is equivalent to annihilating it as a science, and as a conscious activity in general. Even in higher animals, playing with various things is directed to some functionally justifiable goal—the more so with humans, who first become aware of the goal, and then explore the ways to it. Zenkin forgot to list yet another great philosopher who dealt with the problem of infinity in mathematics and related it to the principles of dialectical logic—this philosopher must certainly be well known to Zenkin, and his name is Karl Marx. He wrote that the poorest architect differs from the most perfect bee in that he has the whole construction in his head before actually constructing it—exactly like a mathematician has an idea of a real number before constructing it from any other entities, and the idea of infinity before delineating the ways of approaching it. Cantor could do that; if Zenkin cannot, this is not a personal trait to much boast about.

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January 4, 2001